Shadowing of Weakly Pseudo-Hyperbolic Pseudo-Orbits in Discrete Dynamical Systems

by

Stephen Thomas Gaito

A thesis submitted to the
Mathematics Institute at the University of Warwick
in conformity with the requirements for
the degree of Doctor of Philosophy.

University of Warwick
Coventry, United Kingdom

May 18, 1992

Abstract

We consider C^r ($r \geq 1 + \gamma$) diffeomorphisms of compact Riemannian manifolds. Our aim is to develop the analytic machinery required to describe the topological symbolic dynamics of sets of weakly hyperbolic orbits. The Pesin set is an example of such a set.

For Axiom-A dynamical systems, that is, for diffeomorphisms which have a uniformly hyperbolic nonwandering set which is the closure of the periodic orbits, this analytic machinery is provided by the Shadowing Lemma. This lemma is a consequence of the Stable Manifold Theorem, and the local product structure of the nonwandering set of an Axiom-A diffeomorphism.

Weakly hyperbolic invariant sets, such as the Pesin set, do not, in general, have local product structure. We can however, prove a generalization of the Shadowing Lemma by combining Anosov's Stability Lemma with the Stable Manifold Theorem. In essence we prove a perturbed Stable Manifold Theorem. In order to deal with weakly hyperbolic orbits we use Pugh and Shub's graph transform version of Pesin's Stable Manifold Theorem.

Normally, the contraction required to prove either Anosov's Stability Lemma or the Stable Manifold Theorem, is derived from the hyperbolicity of a "supporting" invariant set. In fact neither of these proofs require this invariance; hyperbolic, or even pseudo-hyperbolic, families of pseudo-orbits are all that they require. This allows us to conclude the existence of shadowing orbits in the neighbourhood of "hyperbolic invariant sets" of numerical simulations of low-dimensional dynamical systems. In particular corresponding to any such numerical "hyperbolic invariant set", there is a uniformly hyperbolic invariant set of the dynamical system itself.

Acknowledgments

I wish to thank my supervisor, Dr. Anthony Manning, for the help and encouragement he has given me over the past few years. I would also like to thank both Prof. James Eells and especially Dr. Mario Micallef for the *many* hours spent helping me understand the finer points of Riemannian and Finsler Geometry... even though most of this knowledge has not found its way into *this* thesis. I am grateful to both the Nonlinear Systems Laboratory at the Mathematics Institute of the University of Warwick, and to NSERC (Canada) for the financial assistance they have extended to me while I have been conducting the research for this Doctorate. Last, but very far from least, I would like to thank Maureen Greyson and our daughter, Hannah Greyson-Gaito, for remaining constant companions throughout the many long hours which have been needed to write this thesis.

Declaration of Authorship

The contents of this thesis have never appeared before in any published format. Moreover, except as noted within, the contents of this thesis consists of entirely original research conducted solely by the author.

Contents

List of Theorems

List of Figures

Chapter 1

Introduction

General mathematical setting The natural setting for the study of differentiable dynamical systems is as a C^r flow or diffeomorphism on a compact Riemannian m-manifold. The manifold affords us with a general setting for the differentiation and integration implied by the word *differentiable* in the name differentiable dynamical system. The Riemannian metric provides a consistent notion of distance over the manifold.

Associated to any Riemannian manifold are its tangent bundle (TM) and its Grassmannian bundle (GM). The manifold's tangent bundle is essentially a linearized version of manifold. The space of sections of the tangent bundle is a natural Banach space associated with manifold. By using the exp map we can lift the action of any diffeomorphism on the manifold into an action on the sections of the tangent bundle. We can, in this way, translate an inherently non-linear problem into, an infinite dimensional, linear problem.

The Grassmannian bundle is the natural collection of subspaces associated to the manifold's tangent bundle. That is, the fibres of the Grassmannian bundle consist of the collection of all of the k planes ($0 \leq k \leq m$) of the corresponding fibre of the tangent bundle. The Grassmannian bundle is then the appropriate object to use to discuss relationships between subbundles of the tangent bundle.

For this thesis we are exclusively interested in dynamical systems which result from the (discrete) action of a diffeomorphism on a manifold. We will normally consider one fixed dynamical system which will consist of a given C^∞ finite m-dimensional compact manifold M, a given C^∞ Riemannian metric on M, and a given C^r ($r \geq 1 + \gamma$) diffeomorphism f of M to itself.

Our main results: The Weak Shadowing Stable Manifold Theorem In the study of dynamical systems two dynamical objects standout as being quite important, namely invariant sets and pseudo-orbits [1]. In fact a pseudo-orbit is a generalization of a one orbit invariant set. An orbit represents, in a sense, the "actual" behaviour of a point in the dynamical system. A pseudo-orbit represents one possible result of a numerical simulation of the behaviour of *a* point in the dynamical system.

Not surprisingly there is a relationship between these two types of objects. An orbit can *shadow* a pseudo-orbit. In fact one of the most important theorems of uniformly hyperbolic dynamical systems, the Shadowing Lemma, states that any (appropriate) pseudo-orbit of a uniformly hyperbolic set is shadowed by an unique orbit of the system. More precisely

Lemma 1.1 *Let Λ be a uniformly hyperbolic closed f-invariant subset of M. Then there exists an open neighbourhood U of Λ and a neighbourhood V of f in $Diff^1(M)$ such that for any $\beta > 0$ there exists an $\alpha > 0$ such that any g in V and any α-pseudo g-orbit contained in U is β-shadowed by a unique g-orbit. Moreover, this orbit is uniformly hyperbolic, and, if $g = f$ and Λ is locally maximal, then the orbit is contained in Λ.*

One of the important conclusions of the Shadowing Lemma is that to any appropriate numerical simulation of a uniformly hyperbolic system there corresponds a "real" orbit of the system which behaves in precisely the way that the simulated orbit behaved.

An important, theoretical, consequence of the Shadowing Lemma, is that if we "know" the behaviour of all of the pseudo-orbits formed of a small (finite) subset of the uniformly hyperbolic set then we "know" the behaviour of the uniformly hyperbolic set itself. This is precisely the content of Bowen's proof [Bow75] that every locally maximal uniformly hyperbolic set is the finite-to-one image of a shift of finite type. That is, the action of the shift of finite type is (semi)conjugate to the action of the diffeomorphism f restricted to the hyperbolic set.

[1] All of the relevant concepts used in this section will be defined in the following sections.

Unfortunately uniformly hyperbolic dynamical systems are not generic. That is, it is not true that "most" dynamical systems are uniformly hyperbolic. The aim of this thesis is to extend the Shadowing Lemma to non-uniformly hyperbolic dynamical systems.

For uniformly hyperbolic dynamical systems, there are two different proofs of the Shadowing Lemma. The first proof, due to Bowen [Bow75], relies on the *local product structure* of a uniformly hyperbolic invariant set. Unfortunately non-uniformly hyperbolic invariant sets do not, in general, have local product structure. This makes it difficult to generalize Bowen's proof[3].

It is the other proof of the Shadowing Lemma, due to Anosov [Ano70, Kat81], which we will generalize to non-uniformly hyperbolic dynamical systems. Anosov's proof is essentially an application of the Contraction Mapping Principle.

One of the most important assertions of the Shadowing Lemma for locally maximal uniformly hyperbolic dynamical systems, is that the orbit, which shadows the original pseudo-orbit, is itself a member of the original uniformly hyperbolic invariant set. For a non-uniformly hyperbolic dynamical system we can not, generally, make such a strong assertion. In the non-uniformly hyperbolic case the best we can do is to assert that the shadowing orbit of appropriately chosen pseudo-orbits will "lose" an arbitrarily small "amount" of hyperbolicity.

In order to provide the required hyperbolicity estimates for the shadowing orbit, we will replace Anosov's application of the Contraction Mapping Theorem by an application of a modified form of Pugh and Shub's version of Pesin's, non-uniformly hyperbolic, Stable Manifold Theorem for vector Bundles [PS89]. This method of proof has the effect of unifying the Stable Manifold Theorem and the Shadowing Lemma. The Shadowing Lemma is really a perturbed Stable Manifold Theorem; the Stable Manifold Theorem is an unperturbed Shadowing Lemma.

In the Shadowing Lemma, it is the hyperbolicity of the hyperbolic invariant set Λ, which provides the contraction required to apply the Contraction Mapping Theorem which in turn guarantees the existence of the shadowing orbit. The "shadowing part" of the proof of the Shadowing Lemma does not require the set Λ to be invariant. We will prove the Shadowing Lemma for a pseudo-hyperbolic

[3] For an example of a generalization of Bowen's proof see [KT92]

family of pseudo-orbits. With this generalization, our version of the Shadowing Lemma can then be applied to numerically verified "hyperbolic invariant sets" of a numerical simulation.

Roughly the statement of our version of the Shadowing Lemma is the following. Given a pseudo-hyperbolic invariant set $\mathfrak{A}$, there exists a neighbourhood $\boldsymbol{U}$ of $\mathfrak{A}$ for which any appropriately chosen pseudo-orbit, $\mathfrak{B}$, contained in $\boldsymbol{U}$ is uniquely shadowed by a (weakly) hyperbolic invariant set, $\mathfrak{C}$. The difference in the amount of hyperbolicity of the pair of invariant sets $\mathfrak{A}$ and $\mathfrak{C}$ can be made arbitrarily small, and moreover the hyperbolic invariant set $\mathfrak{C}$ has C^r local stable and unstable manifolds.

In fact we prove more than this. The "Shadowing Lemma" which we have sketched above is what we call the Weak Shadowing Stable Manifold Theorem. This theorem is stated and proven in chapter 14. It is in turn a consequence of the four main theorems contained in chapters 10, 11, 12 and 13.

To state these four theorems we need to introduce some new concepts. The main interest of our theory is to obtain shadowing invariant sets for weakly hyperbolic pseudo-orbits. The main property of a weakly hyperbolic pseudo-orbit is its division into a countable collection of (possibly non-f-invariant) subsets over which the hyperbolicity conditions *are* uniform. We call these subsets hyperbolic blocks. Corresponding to this division into a countable collection of hyperbolic blocks it is natural to construct a paracompact manifold, $\boldsymbol{M}$, which is the disjoint union of a countable collection of copies of M. Given any pseudo-orbit, $\mathfrak{A}$, of M we can "lift" it to a "pseudo-orbit", $\boldsymbol{\mathfrak{A}}$, of $\boldsymbol{M}$.

Roughly the four theorems, which form the content of the proof of the Weak Shadowing Stable Manifold Theorem, state the following facts:

1. To any (appropriate) weakly hyperbolic pseudo-orbit, $\mathfrak{A}$, we can associate a a C^∞ Riemannian metric of $\boldsymbol{M}$ with respect to which the lifted pseudo-orbit, $\boldsymbol{\mathfrak{A}}$, is uniformly hyperbolic (Theorem 10.1 chapter 10).

2. To any (appropriate) uniformly pseudo-hyperbolic pseudo-orbit, $\boldsymbol{\mathfrak{A}}$, of $\boldsymbol{M}$ there exists a unique f-invariant splitting with respect to which the pseudo-orbit, $\boldsymbol{\mathfrak{A}}$, is hyperbolic (Lemma 11.1 chapter 11).

3. To any (appropriate) uniformly hyperbolic invariant set, $\mathfrak{A}$, of $\boldsymbol{M}$ there exists an open neighbourhood, $\boldsymbol{U}$, of $\boldsymbol{M}$ for which *any* pseudo-orbit, $\mathfrak{B}$, of $\boldsymbol{M}$ which is contained in the neighbourhood, $\boldsymbol{U}$, is uniformly pseudo-hyperbolic (Lemma 12.1 chapter 12).

4. Any (appropriate) uniformly hyperbolic pseudo-orbit, $\mathfrak{A}$, of $\boldsymbol{M}$ is shadowed by a hyperbolic invariant set of $\boldsymbol{M}$ (Theorems 13.1 and 13.2, chapter 13).

Chapters 8 and 9 are devoted to making these rather vague concepts mathematically concrete as well as proving the lemmas required to pass back and forth between pseudo-orbits of M and pseudo-orbits of $\boldsymbol{M}$.

These four main theorems in turn depend on the C^r_κ-section theorem and the Unstable Manifold theorem for vector bundles which are stated and proven in chapters 6 and 7 respectively. Finally, all of the above theory depends on a number of facts and concepts, of a differential geometric nature, which are collected in chapters 2, 3, 4, and 5.

The structure of this thesis In total, the contents of this thesis falls, rather nicely, into four main groups which correspond to Parts I, II, III, and IV of this thesis. Part I contains a collection of differential geometric facts. Part II contains the Cr3κr-section and Unstable Manifold theorem for vector bundles, both of which are primarily vector bundle theorems. Part III contains the definitions of pseudo-hyperbolic pseudo-orbits as well as the four main theorems which are together of most interest to dynamical systems. Part IV contains two examples of the use of the theory developed in Part III: the Weak Shadowing Stable Manifold theorem, and a strengthened version of a theorem by Katok [Kat80]. Our version of Katok's theorem (Corollary 14.3) states that if an f-invariant Borel measure has characteristic exponents which are strictly bounded away from 1, then the support of μ is contained in the closure of the hyperbolic periodic orbits whose hyperbolicity is also similarly strictly bounded away from 1.

Relationship to previous results For anyone who is familiar with the proofs of the uniformly hyperbolic Unstable Manifold Theorem (see [HP70, HPS77, Shu87]), Anosov's Stability Lemma (see [Shu87, Kat81, Ano70], see also [Mat68]), as well as the various proofs of Pesin's non-uniformly hyperbolic Stable Manifold Theorem (see [Pes76, FHY83, PS89]) little in this thesis will be altogether new. The contents of this thesis draws heavily on the ideas contained in the work mentioned above. We have, however, freely rearranged these ideas in order to give primary emphasis to the concept of a pseudo-hyperbolic pseudo-orbit. While the idea of a pseudo-orbit is manifestly present in Anosov's stability lemma (see [Kat81]) it was not identified as a central concept.

The most important *new* idea which we *have* added to this circle of ideas is that it is possible to convert a *weakly* hyperbolic pseudo-orbit on a compact manifold into a *uniformly* hyperbolic pseudo-orbit on a paracompact manifold. Since the "Graph Transform" method of proving the Unstable Manifold theorem only requires appropriate uniformities and *not compactness* of the manifold, we can use it to replace the rather cruder fixed point argument in Anosov's Stability Lemma to essentially obtain our Weak Shadowing Stable Manifold theorem.

Instead of directly reproving Anosov's Stability Lemma we have chosen to break it down into a number of individually interesting parts. These parts consist of the definition of a pseudo-hyperbolic pseudo-orbit, and the four main theorems referred to above (Theorem 10.1, Lemma 11.1, Lemma 12.1, and Theorems 13.1 and 13.2). Indeed, the whole of Part III represents a reworking of Anosov's Stability Lemma.

The C^r_*-Section theorem and the Unstable Manifold theorem as proven in Part II are essentially *not* new. The C^r_*-Section theorem can be found in Pugh and Shub's paper [PS89]. We have chosen to reprove this theorem in this thesis because parts of the arguments are similar to those used in Part III. While the proof of the Unstable Manifold theorem given in this thesis does not exist in the literature, much of our proof relies heavily on the proofs given by Shub in [Shu87] and sketched in Pugh and Shub's paper [PS89]. We give this reproof for two reasons. Firstly, we want to check that the old proof still applies to our new conditions; not surprisingly, it does. Secondly, we need to add the required estimates of the hyperbolicity of the intersection of the stable and unstable manifolds. While these latter estimates of hyperbolicity have never appeared in the literature before, this is only due to the fact that the perceived need for these estimates only appears in our rendition of Stability/Shadowing theory.

Finally, the Differential Geometric facts which we collect in Part I come from many diverse sources. Some of these facts, such as the Lipschitz inverse function theorem can be (essentially) found in the literature (see [HP70, Shu87]). Some of these facts, such as the lemmas relating to the non-degeneracy of splittings and metrics for Grassmannian manifolds may or may not be new. We have chosen to prove what we felt was required to prove the main results in this thesis. Some of the facts, such as the definitions of normed bundles and their constructions can definitely be found elsewhere (see [Hir76, AMR83, Hus75]).

Reading this thesis While we have placed the chapters and the major Parts in the order of logical dependency, this is *not* the best order in which to read the thesis. We have provided both an index and a table of symbols in order to allow the reader to read the thesis from the back to the front. Read this way, the reader will be more able to understand why the previous definitions and lemmas have been included in this thesis. The *best* chapter to read first is chapter 14 of Part IV. Reading this chapter first will give the reader the flavour of our theory and what can be proven with it. Having read Part IV, the reader should then read Part III in order. Parts I and II need only be read as and when the reader feels the need.

Results left out of this thesis Finally, this thesis represents only part of the full thesis which we could have submitted. Most of the results which can be found in uniformly hyperbolic stability theory (as defined by, for example, Shub's book [Shu87]) are now present in what we could call weakly hyperbolic stability theory (essentially this thesis). There are two notable exceptions. Firstly, proofs of structural stability which can be conducted for uniformly hyperbolic dynamical systems are not presented here. Indeed it is likely that they can *not* be proven for weakly hyperbolic dynamical systems.

Secondly, as part of uniformly hyperbolic stability theory, it is relatively easy to show that the f-invariant splittings with respect to which an orbit is uniformly hyperbolic are Hölder continuous. We *can* "relatively" easily make similar weakly hyperbolic statements. Namely, there exists a pair of sequences, (γ_n) and (K_n), which are controllably decreasing and increasing respectively, for which the splitting over the n^{th} hyperbolic block is continuous with respect to a γ_n-Hölder norm and, moreover, this γ_n-Hölder norm is K_n bounded over the n^{th} hyperbolic block.

Unfortunately, to make this statement we require a "norm-like" metric in the tangent bundle, $\boldsymbol{TM}$, of $\boldsymbol{M}$. For a Riemannian geometer the most obvious metric in the tangent bundle is the Sasaki metric (see [Sas58, Dom62]) unfortunately the induced metric in each fibre does *not* correspond to the norm induced metric in each fibre. This fact makes the Sasaki metric inappropriate for our use. The previous work in uniformly hyperbolic dynamical systems has used "admissible" metrics. These admissible metrics are essentially embeddings of the Riemannian manifold in a Euclidean space of some large dimension. Again these metrics are inappropriate for our use because we need not one but a countably infinite number of such embeddings. Since we have little control over these embeddings we have little control over the metric.

Fortunately, we *have* found a suitable "metric" of $\boldsymbol{TM}$ with which we can prove the required results. Unfortunately, this "metric" is only locally a metric. Since such objects are not well known, if at all, their definition and the proofs of their basic properties are quite lengthy. This length would have considerably increased the length of this, already long, thesis. Since it is not obvious that our particular "metric" is indeed the most appropriate one, we have decided to leave this result out of this thesis. We will to leave the statement and proof of these Hölder results for the time, in the near future, when the need of these results is more directly obvious.

Part I

Bundles, bundle constructions, and the continuity of bundle maps.

In Part II of this thesis we prove the perturbed Stable Manifold theorem for disc bundles. The central idea behind this proof is really very simple: recursively apply the contraction mapping principle uniformly in each fibre to an appropriately constructed pair of vector bundle morphism and normed (Finsler) vector Bundle. The technique is contained in constructing the correct vector bundle morphism, bundle pair at each step. The purpose of this part of the thesis is to collect, in one place, all of the *vector bundle* constructions which will be required in both Parts II and III.

The objective of applying the contraction mapping principle in each fibre, in a given vector bundle, is to show that there exists a unique section of the vector bundle which is invariant under the action of the bundle morphism. This invariant section is then used to construct the next bundle morphism, bundle pair for the next step in the recursion.

In order to apply the contraction mapping principle at all, we require a consistent notion of norm in the fibres of the vector bundle. The most important part of each bundle construction is to show that the system of norms in the fibres of the constituent bundles induces a system of norms in the fibres of the constructed bundle.

In the first chapter of Part I, we define the concepts of Metric and Normed vector bundles, the space of sections of these bundles. In the second chapter of Part I, we collect all of the bundle constructions which will be require in this thesis.

The work in Part III will require a detailed understanding of the continuity of splittings of TM. The most natural object in which to study this continuity is the Grassmannian bundle associated to TM. The third chapter is devoted to a discussion of this bundle.

We collect in the last chapter of Part I, all of the Lipschitz continuity results that we will need in the rest of the thesis.

Chapter 2

Elementary Differential Geometry

Our primary interest in this chapter is to briefly review the definitions and results from elementary Differential geometry which we will require throughout the rest of this thesis. Unless noted otherwise, all of these definitions and results are taken from one or other of the following texts: [AMR83, Hir76, Hus75] (for differential geometry), [Hel78, SGL90] (for Riemannian geometry) and [Run59, Bej90] (for Finsler geometry).

2.1 Metrics

It will be convenient, for the purposes of this thesis, to extend the definition of a metric in a number of different ways. Let $\mathbf{R}_\infty$ denote the *extended* real line, $\mathbf{R} \bigcup \{\pm\infty\}$, then a K-metric of a topological space X, is a function $d : X \times X \rightarrow \mathbf{R}_\infty$ for which for any $x, y, z \in X$ we have

- **Positive:** $d(x, y) \geq 0$,
- **Definiteness:** $d(x, y) = 0$ iff $x = y$,
- **Symmetry:** $d(x, y) = d(y, x)$, and
- **r-local K-triangle inequality:** $d(x, z) \leq Kd(x, y) + Kd(y, z)$.

Note that a 1-metric corresponds to the normal definition of a metric.

Note that with this definition of a metric, pairs of points, in X, can be an infinite distance apart. This will be useful when we consider the metric of a disjoint union of metric spaces. In this case two points in disjoint components of the disjoint union will be defined to be an infinite distance apart. Having made this extended definition of a metric, it will also be convenient, unless stated otherwise, to only consider arbitrary points x and y of a metric space X which are a *finite* distance apart.

A pair of metrics, $d_1(\cdot,\cdot)$ and $d_2(\cdot,\cdot)$, are c_1-c_2-*topologically equivalent* if for all $x, y \in X$

$$\frac{d_1(x,y)}{c_1} \leq d_2(x,y) \leq c_2 d_1(x,y).$$

In particular, $d_1(x,y) = \infty$ iff $d_2(x,y) = \infty$. A pair of metrics are c-*topologically equivalent* if they are c-c-topologically related.

With these definitions, it is easy to show that, if a symmetric positive definite function, $\bar{d}(\cdot,\cdot)$, is c-topologically equivalent to a K-metric, $d(\cdot,\cdot)$, then the symmetric positive definite function, $\bar{d}(\cdot,\cdot)$, is a c^2K-metric.

2.2 Bundles

A *bundle*, $\pi : \boldsymbol{E} \rightarrow B$, is a pair of topological spaces , $\boldsymbol{E}$, and B, called the total and base spaces respectively, together with a projection, π. In this thesis, the base space of *every* bundle, will be a metric space. We call this metric the *base metric* of the bundle $\boldsymbol{E}$. In order to unify the discussion of the C^r properties of any given bundle, we explicitly consider a metric space to be a C^0 *manifold.*

A *bundle map* is map between a pair of bundles which preserves the respective fibres. Associated to any bundle map, $\boldsymbol{f}$, is a well defined base map, f, which makes the following diagram commute

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"'] \\ B_E \arrow[r, "f"] & B_F \end{tikzcd}$$

2.3 Metric bundles

A *metric bundle* is a triple, (X, π, A), where X and A are complete metric spaces, and π is a continuous projection, $\pi : X \to A$, with respect to which the fibres of X, $X_a = \pi^{-1}(a) \subset X$ for $a \in A$, are complete metric subspaces of X. Since X is a metric space, there is a globally defined metric. The metric in each fibre is the restriction to the fibre of the metric of the total space, X.

A Metric bundle morphism, is any continuous map between Metric bundles which commutes with the respective projections. That is, a Metric bundle morphism is any continuous map which preserves fibres

2.4 Vector bundles

A C^r *vector bundle*, $\pi : E \to B$, is a locally trivial bundle whose typical fibre, E, is a k-dimensional vector space, whose base space B is a C^r manifold, and whose structure group is the general linear group, $\mathbf{GL}(E)$, of the vector space E.

The "local triviality" means that there exists an atlas of C^r local trivializations (charts), (U_i, ϕ_i), where $U_i \subset B$ and $\phi_i : \pi^{-1}(U_i) \to U_i \times E$. Moreover, for any given pair of overlapping trivializations, (U_i, ϕ_i), and (U_j, ϕ_j) for which $U_i \bigcap U_j \neq \emptyset$, the overlap or transition maps,

$$\begin{aligned} \psi_{ij} &: U_i \bigcap U_j \times E \to U_i \bigcap U_j \times E, \\ \psi_{ij} &= \phi_i \circ \phi_j^{-1}, \end{aligned}$$

are C^r.

The fact that the structure group of the vector bundle is $\mathbf{GL}(E)$, means that the tangent map, $T\psi_{ij}$, to the overlap maps, ψ_{ij}, are when restricted to each fibre of TE, elements of the structure group, $\mathbf{GL}(E)$.

A *vector bundle map* is map between a pair of vector bundles which preserves the respective fibres *and* is a linear map when restricted to any one fibre.

2.5 Orthogonal vector bundles, and Riemannian manifolds

A C^r *orthogonal bundle* is a C^r vector bundle whose structure group is the set of orthogonal (isometric) vector space morphisms, $\mathbf{O}(k, \mathbf{R})$, of the vector space E. An *orthogonal structure* for a vector bundle, is a C^r section of the tensor bundle, $\mathbf{T}^0_2(\boldsymbol{E})$, which defines an inner product, and hence a norm, in each fibre of $\boldsymbol{E}$. Every orthogonal vector bundle is equipped with a unique orthogonal structure for which the chart maps restricted to each fibre, $\phi_{ix} = \phi_i\big|_{\boldsymbol{E}_x}$, are isometric with respect to the induced inner product. Conversely, every orthogonal structure on a vector bundle, induces a unique (complete) orthogonal atlas. See [Hir76][page 95] or [Hus75][Chapter 5 section 7].

Note that except for the requirement that the inner product or norm vary continuously (C^r) from fibre to fibre, this definition does not imply any consistency in the action of the inner product or norm *between* fibres. In order to stress this fact we will often denote which fibre a given norm is taken in by writing $|e| = |e|_{\pi(e)}$ for $e \in \boldsymbol{E}$. We call these norms in the fibres the *fibre norms*.

A C^∞ *Riemannian manifold* is a C^∞ manifold whose tangent bundle is equipped with a C^∞ orthogonal structure.

For our purposes, one of the most important properties of a compact Riemannian manifold is the existence of a strictly positive constant, r, called the injectivity radius for which for any $x \in M$ the exponential map $\exp_x : T_xM \to M$ is a C^∞ diffeomorphism of the ball of radius r about the origin and moreover

$$|v|_x = d(x, \exp_x(v))$$

for all $v \in B_r(0)$. We will use $\exp_x^{-1}$ as a convenient way to locally lift the action of a C^1 diffeomorphism, $f : M \to M$, into a suitable disk subbundle of the tangent bundle, TM.

2.6 Normed vector bundles, and Finsler manifolds

While every orthogonal vector bundle is equipped with a globally defined norm, not every vector bundle which is equipped with a globally defined norm is an orthogonal vector bundle. While it was probably not Riemann's original intention to restrict his attention to norms defined by an orthogonal structure, it was Paul Finsler, in 1918, who is credited as being the first person to systematically study normed vector bundles which are not induced by an orthogonal structure.

A C^r *normed vector bundle*, $\boldsymbol{E}$, is a C^r vector bundle which is equipped with a globally defined function, $L : \boldsymbol{E} \rightarrow \mathbf{R}$ which is C^r except on the zero section of $\boldsymbol{E}$, and which, when restricted to any fibre of $\boldsymbol{E}$, is a vector space norm of the fibre. The function, L, is often called the fundamental function, or norm, of the normed vector bundle $\boldsymbol{E}$.

For $x \in B$, let L_x denote the function $L_x : \boldsymbol{E}_x \rightarrow \mathbf{R}$. A C^r *Finsler bundle*, $\boldsymbol{E}$, is a C^r normed vector bundle which satisfies the additional regularity condition: the quadratic form $D^2\left(L_x^2(v)\right)(v_1, v_2)$ is positive definite for all $v, v_1, v_2 \in \boldsymbol{E}_x$. Rund [Run59] shows that this additional condition implies that L_x satisfies the triangle inequality. Conversely he gives an example of a norm for which the above quadratic form is not positive definite.

A C^r *Finsler manifold* is a C^r manifold whose tangent bundle is a *Finsler* vector bundle. For a Finsler manifold the additional regularity condition, given above, is equivalent to a Legendre condition. This condition is required in order to ensure that the calculus of variation line integral map

$$\int_0^1 L\left(c'(t)\right) dt$$

defined on the space of piecewise C^1 curves, $c : [0,1] \rightarrow M$, has well defined, minimal, extremals. This in turn implies that there is a well defined distance metric defined by the infimum of the lengths of all piecewise C^1 curves joining a given pair of points x and y in M. Rund's book [Run59] has a more complete discussion of this topic.

Though we will not proof this claim, we claim that the theory which we will develop in this thesis most naturally, resides in the category of Finsler manifolds. None of the arguments contained in this thesis *require* the existence of an inner product in the fibres and so none of the arguments require any bundle to have an orthogonal structure. However since the exponential map of a Finsler manifold is only C^1 on the zero section[1] there is no *convenient* way to locally lift the C^1 diffeomorphism $f : M \to M$. While any collection of local diffeomorphisms from M into the fibres of TM could be used to locally lift the action of f into TM, the norm of the lift would no longer be simply related to the distance metric in the manifold. To simplify the arguments in this thesis we will consequently restrict our attention to the category of Riemannian manifolds.

Finally, in order to complete our notation, consider a normed vector bundle, $\pi : \boldsymbol{E} \to B$, and a continuous function $r : B \to (0, \infty)$. We define the *varying disc bundle* for $\boldsymbol{E}$ to be $\triangle_r \boldsymbol{E} = \left\{ e \in \boldsymbol{E} \mid |e|_{\pi(e)} \leq r(\pi(e)) \right\}$.

When dealing, as above, with a function defined on the base space, B, of any bundle, we will implicitly extend the definition to the whole of the total space of the bundle, $\pi : \boldsymbol{E} \to B$, via the pull back by the projection, π. That is for any function, r, defined on B, we will define, $r = \pi^* r = r \circ \pi$, on $\boldsymbol{E}$.

2.7 Spaces of bundle maps

Given a pair of vector bundles, $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, we denote the set of all *vector* bundle maps between them by $\mathbf{L}(\boldsymbol{E}_1, \boldsymbol{E}_2)$. If the $\boldsymbol{E}_i$ are normed bundles this set, $\mathbf{L}(\boldsymbol{E}_1, \boldsymbol{E}_2)$, is a Banach space. For a vector bundle map, $F \in \mathbf{L}(\boldsymbol{E}_1, \boldsymbol{E}_2)$, the norm of F, $|F|$, is defined by the supremum over the base space of $\boldsymbol{E}_1$ of the operator norms of each linear fibre maps of F. That is

$$|F| = \sup_{x \in B_1} \left\| F|_{\pi_1^{-1}(x)} \right\|.$$

Given a pair of *metric* bundles $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ we denote the set of all metric bundle maps between them by $C^0(\boldsymbol{E}_1, \boldsymbol{E}_2)$. This set is a metric space where the metric is the sup metric defined by the fibre metrics of the bundle $\boldsymbol{E}_2$. If $\boldsymbol{E}_2$ is

[1] See Rund's discussion of normal coordinates for Finsler manifolds [Run59]

a normed bundle, then $C^0(E_1, E_2)$ is a Banach space with the sup norm defined by the fibre norms of E_2.

Since a normed trivial vector bundle, E, has a well defined metric on its total space which is, when restricted to a fibre, the metric induced by the norm in that fibre, the bundle E is also a metric bundle whenever we choose to forget the vector space structure of each fibre. This means that any vector bundle map between normed vector bundles is also a metric bundle map between the same bundles when considered as metric bundles. Hence $\mathrm{L}(E_1, E_2) \subset C^0(E_1, E_2)$ *as sets.* However the Banach space of vector bundle maps is *not* a subspace of the Banach space of metric bundle maps into E_2 because the respective norms are different.

2.8 The Graph transform and convergence in the space of sections of a bundle

Given a bundle, $\pi : E \to B$, we can consider the *space of sections*, $\Gamma(E)$, of E. Recall that a *section* of E is defined to be a map $\sigma : B \to E$ for which $\pi \circ \sigma = Id_B$. If E is a Metric bundle, we use the sup metric,

$$d(\sigma, \bar{\sigma}) = \sup_{x \in B} \eth(\sigma(x), \bar{\sigma}(x)),$$

to define a metric on $\Gamma(E)$. If E is a normed vector bundle, we use the sup norm,

$$|\sigma| = \sup_{x \in B} |\sigma(x)|_x,$$

to define a metric on $\Gamma(E)$. In both cases, since the fibres of E are assumed to be complete, the metric (norm) on $\Gamma(E)$ ensure that $\Gamma(E)$ is itself a complete metric (Banach) space.

If the base space, B, of a vector bundle, E, is a vector space then we can consider the Banach space of all *linear* sections of E. We denote this space by the symbol $\Gamma_{\mathrm{L}}(E)$. The norm in this case is the operator norm

$$|\sigma| = \sup_{\{x \in B \mid |x| = 1\}} |\sigma(x)|_x .$$

If the base space, B, is itself a vector bundle, over the space H, then we can consider the space of all sections of $\boldsymbol{E}$ which are linear in each *fibre* of the vector bundle B. In this case the Banach space norm is the sup norm of the operator norms,

$$\begin{aligned} |\sigma| &= \sup_{x\in H} \left\| \sigma|_{B_x} \right\| \\ &= \sup_{x\in H} \sup_{\{b\in B \mid |b|_x=1\}} |\sigma(b)|_b . \end{aligned}$$

Now consider a pair of bundles (metric or normed vector bundle), $\pi_E : \boldsymbol{E} \longrightarrow B_E$ and $\pi_F : \boldsymbol{F} \longrightarrow B_F$ and an appropriate bundle map, $f : \boldsymbol{E} \longrightarrow \boldsymbol{F}$, between $\boldsymbol{E}$ and $\boldsymbol{F}$ whose base map, $f_0 : B_E \longrightarrow B_F$, is a C^r diffeomorphism (C^0 homeomorphism). Such a bundle map induces a unique map, $\Gamma_f : \Gamma(\boldsymbol{E}) \longrightarrow \Gamma(\boldsymbol{F})$, between the spaces of sections of $\boldsymbol{E}$ and $\boldsymbol{F}$, defined by

$$\Gamma_f(\sigma) = f \circ \sigma \circ f_0^{-1}.$$

This map, Γ_f, on the space of sections is as differentiable as the map, f, on the bundles, which is in turn no more differentiable then the base map, f_0.

We note that Pugh and Shub, in their paper [PS89], use scaled fibre norms on disc bundles, $\triangle_r\boldsymbol{E}$. For $e \in \triangle_r\boldsymbol{E}$ they define a *scaled* fibre norm,

$$|e|_x^* = \frac{|e|_x}{r(x)}.$$

This scaled fibre norm induces a correspondingly scaled norm for the space of sections. They comment that C^1 convergence with respect to the scaled norm implies C^1 convergence with respect to the unscaled norm [PS89][page 16]. This implicitly assumes that convergence with respect to the scaled norm implies convergence with respect to the unscaled norm. While the manifold M which we consider is compact, we will typically consider subsets of M which are only σ-compact. In this case convergence with respect to the scaled norm *does not* imply convergence with respect to the unscaled norm. This fact, due to the non-compactness of the sets we consider is best seen by the following pair of simple counterexamples.

Consider the trivial bundle $\boldsymbol{E} = \mathbb{N} \times \mathbb{R}$.

First consider the function $r : \mathbb{N} \to \mathbb{R}^+$ defined by $r(n) = n$, let $\sigma(n) = \frac{1}{2}$, and for $m \in \mathbb{N}$ let

$$\sigma_m(n) = \begin{cases} \frac{1}{2} & \text{if } n \le m \\ 0 & \text{otherwise,} \end{cases}$$

then both σ and σ_m are sections of $\triangle_r \boldsymbol{E}$.

$$|\sigma_m(n) - \sigma(n)| = \begin{cases} 0 & \text{if } n \le m \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

$\sigma_m \to \sigma$ with respect to the scaled norm but not with respect to the unscaled norm.

Now consider the function $r : \mathbb{N} \to \mathbb{R}^+$ defined by $r(n) = \frac{1}{n}$, let $\sigma(n) = \frac{1}{2n}$, and for $m \in \mathbb{N}$ let

$$\sigma_m(n) = \begin{cases} \frac{1}{2n} & \text{if } n \le m \\ 0 & \text{otherwise,} \end{cases}$$

then both σ and σ_m are again sections of $\triangle_r \boldsymbol{E}$. Since

$$|\sigma_m(n) - \sigma(n)| = \begin{cases} 0 & \text{if } n \le m \\ \frac{1}{2n} & \text{otherwise,} \end{cases}$$

$\sigma_m \to \sigma$ with respect to the unscaled norm but not with respect to the scaled norm.

Chapter 3

Metric and normed vector bundle constructions

It is the purpose of this chapter to show that all of the bundle constructions required in Parts II and III are natural constructions in the category of Metric and normed vector bundles. In each construction, we will explicitly verify that the construction is natural in the category of normed vector bundles, we leave the corresponding verification for the category of Metric bundles to the reader.

In addition, for all of the constructions, given below, we state without proof, that if each of the constituent bundles have a pair of fibre norms and a corresponding pair of base metrics which respectively satisfy c-topological equivalence relations of the type

$$\begin{aligned} \frac{1}{c}|\cdot|. \leq \quad & |\cdot|. \quad \leq c|\cdot|., \quad \text{and} \\ \frac{1}{c}d(\cdot,\cdot) \leq \quad & d(\cdot,\cdot) \quad \leq cd(\cdot,\cdot), \end{aligned}$$

then the constructed bundle has a similar pair of norms and base metrics also satisfy c-topological equivalence relations *with the same constant c*. Again, we leave without proof the similar statement for the category of Metric bundles.

3.1 Disjoint unions

Given a weakly hyperbolic invariant set, Λ, of the diffeomorphism f on a Riemannian Manifold M, we will construct a countable collection of norm structures on the manifold M, called the Pesin-Mather norms, with respect to which the action of f on the invariant set Λ *looks* uniformly hyperbolic. The most convenient way to work with this countable collection of Riemannian metrics is by considering it as a *single* norm defined on the disjoint union of a countable collection of copies of the manifold M. In this construction, each copy of M is equipped with exactly one of the Pesin-Mather norms and moreover every Pesin-Mather norm is carried on exactly one copy of M.

Let A be an index set, and let $\pi_\alpha : \boldsymbol{E}_\alpha \to B_\alpha$ be a collection of C^r normed vector bundles for each $\alpha \in A$. For simplicity we assume that all of the constituent vector bundles have the same typical fibre E. Let $B = \bigsqcup_{\alpha \in A} B_\alpha$, $\boldsymbol{E} = \bigsqcup \boldsymbol{E}_\alpha$, and let $\pi = \bigsqcup \pi_\alpha$.

We claim that $\pi : \boldsymbol{E} \to B$ is a C^r normed vector bundle. We define the fibre norms on $\boldsymbol{E}$ as well as the base metric on B to be the disjoint unions of the respective metrics and norms of B_α and $\boldsymbol{E}_\alpha$. In particular, we define the base metric distance between two disjoint components of B to be infinite, that is

$$
\begin{aligned}
d(x,y) &= \begin{cases} d_\alpha(x,y) & \text{if } \exists \alpha \in A \text{ for which } x, y \in H_\alpha, \\ \infty & \text{otherwise,} \end{cases} \\
|e|_x &= |e|_{(x,\hat{\alpha})},
\end{aligned}
$$

where $e, \tilde{e} \in \boldsymbol{E}$, $x, y \in B$, $x = \pi(e)$, $x \in B_{\hat{\alpha}}$, and $|e|_{(x,\hat{\alpha})}$ is the norm of e in $\boldsymbol{E}_{\hat{\alpha}}$.

Similar arguments easily verify that the disjoint union of metric bundles is itself a metric bundle. Since a metric space is a trivial bundle over the one point space, $\{*\}$, the disjoint sum of a collection of metric spaces, is a metric bundle over the index space A taken with some given metric. This is exactly how the metric bundle, which we consider in this thesis, is constructed.

3.2 Pull backs

Given an f-invariant subset U of the manifold M, we can consider the pull back of TM via the canonical inclusion $i : U \to M$. A uniformly hyperbolic f-invariant set Λ is by definition a subset of the manifold M for which the action of Tf on the pull back of TM is (uniformly) hyperbolic. It is a generalization of this construction which is at the heart of our proof of the Weakly Shadowing Stable Manifold theorem given in Part III.

Let $\pi : \boldsymbol{E} \to B$ be a normed vector bundle, and let i be a C^r embedding of a set $\bar{B}$ into B. We will consider the embedding, i, to be a C^r embedding *even if* the map i is only continuous, so long as there exists a (specific) extension of i which is C^r. Consider the i-pull back vector bundle, $\bar{\pi} = i^*\pi : \bar{\boldsymbol{E}} \to \bar{B}$, which is pulled back via the embedding i from the *vector bundle* $\boldsymbol{E}$. Recall that $\bar{\boldsymbol{E}} = \{(\bar{x}, e) \mid i(\bar{x}) = \pi(e)\}$, and that $\bar{\pi}(\bar{e}) = \bar{\pi}(\bar{x}, e) = \bar{x}$ where $\bar{e} = (\bar{x}, e) \in \bar{\boldsymbol{E}}$. The embedding i can be extended to embed the total space of $\bar{\boldsymbol{E}}$ into $\boldsymbol{E}$ via the formula $i(\bar{e}) = i(\bar{x}, e) = e$ where again $\bar{e} = (\bar{x}, e) \in \bar{\boldsymbol{E}}$. Clearly the typical fibre, E, of $\boldsymbol{E}$, is also the typical fibre of $\bar{\boldsymbol{E}}$.

Consider $\bar{x}, \bar{y} \in \bar{B}$, and $\bar{e} \in \bar{\boldsymbol{E}}_{\bar{x}}$. We define the pull back base metric, $\bar{d}(\cdot,\cdot)$, the pull back fibre norm, $\overline{|\cdot|}$, as

$$\begin{aligned} \bar{d}(\bar{x}, \bar{y}) &= d(i(\bar{x}), i(\bar{y})), \quad \text{and} \\ \overline{|\bar{e}|}_{\bar{x}} &= |i(\bar{e})|_{i(\bar{x})}, \end{aligned}$$

In each case, the pull back base metric, or fibre norm, will be as differentiable as either i or its specific extension.

Again, similar arguments show that if $\boldsymbol{E}$ is a Metric bundle and i is continuous, then the pull back of $\boldsymbol{E}$ via i, $\bar{\boldsymbol{E}} = i^*\boldsymbol{E}$, is also a Metric bundle.

3.3 Whitney sum

Both the section theorems and the perturbed Stable manifold theorem, proved in Part II, depend crucially on constructions involving the Whitney sums of metric and normed vector bundles. In particular the objects, which in the bundle versions of these theorems, *correspond* to the stable and unstable manifolds and their higher derivatives *are* sections of the bundle $\bar{\pi}_1$, described below, for some appropriately chosen Whitney sum of bundles.

Consider the following diagram of metric and or normed vector bundles:

$$
\begin{tikzcd}
& \mathbf{E}_1 \oplus_B \mathbf{E}_2 \arrow[dl, "\tilde{\pi}_1"'] \arrow[dr, "\tilde{\pi}_2"] \arrow[dd, "\pi_{12}"] & \\
\mathbf{E}_1 \arrow[dr, "\pi_1"'] & & \mathbf{E}_2 \arrow[dl, "\pi_2"] \\
& B &
\end{tikzcd}
$$

Let $\pi_i : \mathbf{E}_i \to B$, for $i = 1, 2$ be two different C^r normed vector bundles which share the same base space B and base metric $d(\cdot, \cdot)$. Let $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ denote the Whitney sum of the two *vector bundles* $\mathbf{E}_1$ and $\mathbf{E}_2$. Denote a typical element of $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ by $e_1 + e_2$ where $e_i \in \mathbf{E}_i$ and $\pi_1(e_1) = \pi_2(e_2)$. Define the projection $\pi_{12} : \mathbf{E}_1 \oplus_B \mathbf{E}_2 \to B$ by letting $\pi_{12}(e_1 + e_2) = \pi_1(e_1) = \pi_2(e_2)$. Define the projections $\tilde{\pi}_i : \mathbf{E}_1 \oplus_B \mathbf{E}_2 \to \mathbf{E}_i$ by letting $\tilde{\pi}_i(e_1 + e_2) = e_i$. The set $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ can then be viewed as a normed vector bundle in three different ways, over the three different base spaces B, $\mathbf{E}_1$ and $\mathbf{E}_2$.

Consider $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ as a vector bundle over B. Since the $\mathbf{E}_i$ share the same base space and base metric, we can take the base metric of $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ to be the base metric of either $\mathbf{E}_1$ or $\mathbf{E}_2$. We take the fibre norm of $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ to be the box norm constructed out of the fibre norms of the $\mathbf{E}_i$. That is we define

$$|e|_x = \max\left\{ |e_1|_{(1,x)}, |e_2|_{(2,x)} \right\},$$

for $e \in \mathbf{E}_1 \oplus_B \mathbf{E}_2$, $e = e_1 + e_2$, $x = \pi_{12}(e)$ and where, $|e_1|_{(1,x)}$, and $|e_2|_{(2,x)}$, are the respective fibre norms of $\mathbf{E}_1$ and $\mathbf{E}_2$.

The set $\mathbf{E}_1 \oplus_B \mathbf{E}_2$ can also be viewed as a normed vector bundle over either $\mathbf{E}_1$ or $\mathbf{E}_2$. Consider $\mathbf{E}_1$. In this case, the base metric is the total metric of $\mathbf{E}_1$, and the fibre norm is essentially the fibre norm of $\mathbf{E}_2$. That is we define

$$|e|_{e_1} = |e_2|_{(2,x)},$$

for $e \in \mathbf{E}_1 \oplus_B \mathbf{E}_2$, $e = e_1 + e_2$, and $x = \pi_{12}(e)$.

Similar arguments show that $\boldsymbol{E}_1 \oplus_B \boldsymbol{E}_2$ is a C^r normed vector bundle over $\boldsymbol{E}_2$. With these definitions $\boldsymbol{E}_1 \oplus_B \boldsymbol{E}_2$ is then a normed vector bundle over the spaces $\boldsymbol{E}_i$ which are in turn normed vector bundles over B.

If either or both of the constituent vector bundles, $\boldsymbol{E}_1$ or $\boldsymbol{E}_2$, are replaced by Metric bundles, similar arguments can be used to show that $\boldsymbol{E}_1 \oplus_B \boldsymbol{E}_2$ is itself a Metric bundle. If at least one of the $\boldsymbol{E}_i$ is a normed vector bundle then $\boldsymbol{E}_1 \oplus_B \boldsymbol{E}_2$ is a normed vector bundle over the *other* constituent bundle.

In the previous chapter we defined the concept of a varying disc bundle. Consider the pair of continuous functions $r_1, r_2 : B \to (0, \infty)$. We can construct the Whitney sum of the two varying disc bundles $\triangle_r \boldsymbol{E}_1$, $\triangle_r \boldsymbol{E}_2$ which we will denote by $\triangle_{r_1} \boldsymbol{E}_1 \oplus_B \triangle_{r_2} \boldsymbol{E}_2$ and will be called a *doubly varying disc bundle.* This doubly varying disc bundle is a $\hat{r}_i$ varying disc bundle relative to the bundle $\hat{\pi}_i$ where $\hat{r}_i(e_i) = r_i(\pi_i(e_i))$ for $e_i \in \triangle_{r_i} \boldsymbol{E}_i$. We can, in a similar fashion, construct n-varying disc bundles out of n different singly varying disc bundles.

3.3.1 The structure of sections and bundle maps of the Whitney sum

A metric bundle map, F, from any metric or vector bundle, $\bar{\boldsymbol{E}}$, into the bundle π_{12} is a map of the form $F(\bar{e}) = F_1(\bar{e}) + F_2(\bar{e})$ for which F_i is a metric bundle map from $\bar{\boldsymbol{E}}$ into $\boldsymbol{E}_i$. Conversely any pair of metric bundle maps (F_1, F_2) from $\bar{\boldsymbol{E}}$ into the respective bundles $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ can be used to define a metric bundle map from $\bar{\boldsymbol{E}}$ into π_{12} of the above form. This fact is mearly a reflection of the fact that the Whitney sum of vector bundles is the universal product in the category of metric and or vector bundles.

Recall that the sets of metric bundle maps from $\bar{\boldsymbol{E}}$ into respectively π_{12}, and the $\boldsymbol{E}_i$ are denoted by $C^0\left(\bar{\boldsymbol{E}}, \pi_{12}\right)$, and $C^0\left(\bar{\boldsymbol{E}}, \boldsymbol{E}_i\right)$. The previous discussion implies that $C^0\left(\bar{\boldsymbol{E}}, \pi_{12}\right)$ is isomorphic to the product of $C^0\left(\bar{\boldsymbol{E}}, \boldsymbol{E}_1\right)$ and $C^0\left(\bar{\boldsymbol{E}}, \boldsymbol{E}_2\right)$. If both $\boldsymbol{E}_i$ are vector bundles then $C^0\left(\bar{\boldsymbol{E}}, \pi_{12}\right)$ as well as the $C^0\left(\bar{\boldsymbol{E}}, \boldsymbol{E}_i\right)$ are Banach spaces with the appropriate sup norm. Moreover, since the sup norm of $C^0\left(\bar{\boldsymbol{E}}, \pi_{12}\right)$ is derived from the box norm of π_{12}, $C^0\left(\bar{\boldsymbol{E}}, \pi_{12}\right)$ is *isometrically* isomorphic to the product of $C^0\left(\bar{\boldsymbol{E}}. \boldsymbol{E}_1\right)$ and $C^0\left(\bar{\boldsymbol{E}}. \boldsymbol{E}_2\right)$.

Recall that if the $\bar{E}$ and the E_i are normed bundles then the set of *vector* bundle maps from $\bar{E}$ into respectively π_{12}, and the E_i are Banach spaces denoted by $\mathbf{L}\left(\bar{E}, \pi_{12}\right)$, and $\mathbf{L}\left(\bar{E}, E_i\right)$. Again the previous discussion, applied to vector bundles and vector bundle maps, implies that the Banach space $\mathbf{L}\left(\bar{E}, \pi_{12}\right)$ is isometrically isomorphic to the product of the Banach spaces $\mathbf{L}\left(\bar{E}, E_1\right)$ and $\mathbf{L}\left(\bar{E}, E_2\right)$.

If F is a metric bundle map from $\bar{\pi}_1$ to itself, then since it commutes with the projection $\bar{\pi}_1$, the component, F_1 of F, is a metric bundle map from E_1 to itself. Similarly, the component, F_2 of F, is a metric bundle map from $\bar{\pi}_1$ to E_2. We call this latter component, F_2, the *fibre component of F*.

A section, σ, of the bundle $\bar{\pi}_1$ is itself a metric bundle map from E_1 into π_{12} which makes the following diagram commute

$$\begin{tikzcd}
E_1 \arrow[r, "\sigma"] \arrow[d, "\pi_1"'] \arrow[rr, bend left, "Id"] & E_1 \oplus_B E_2 \arrow[r, "\bar{\pi}_1"] \arrow[d, "\pi_{12}"'] & E_1 \arrow[d, "\pi_1"'] \\
B \arrow[r, "Id"] & B \arrow[r, "Id"] & B
\end{tikzcd}$$

Since $\bar{\pi}_1 \circ \sigma = Id_{E_1}$, the section must have the form $\sigma(e_1) = e_1 + s(e_1) \simeq (e_1, s(e_1))$. The bundle map, $s : E_1 \rightarrow E_2$, is called the *fibre component of* σ.

Denote the set of metric bundle maps from E_1 to E_2 whose base maps are the identity, Id_B, by $C^0_{B,Id}(E_1, E_2)$, then this set is isomorphic to the space of sections of the bundle $\bar{\pi}_1$, $\Gamma(\bar{\pi}_1)$. Again more than this is true. Recall that the fibre norm of the bundle, $\bar{\pi}_1$, is essentially the fibre norm of the normed *vector* bundle E_2. This implies that both $\Gamma(\bar{\pi}_1)$ and $C^0_{B,Id}(E_1, E_2)$ are Banach spaces which are isometrically isomorphic with norm

$$|\sigma| = \sup_{e_1 \in E_1} |s(e_1)|_{\pi_1(e_1)}.$$

Note that the set of sections, $\Gamma(\bar{\pi}_1)$, of the bundle $\bar{\pi}_1$ is by definition a subset of the set of metric bundle maps from E_1 into π_{12}, $C^0(E_1, \pi_{12})$ and as such it carries two distinctly different Banach structures. As the subspace of the set of metric bundle maps, $C^0(\boldsymbol{E}_1, \pi_{12})$, the norm is the sup norm formed from the *box* norms of each fibre of π_{12}. As the Banach space of sections of $\bar{\pi}_1$ the norm is the sup norm formed, as above, from the norms of the fibres of $\bar{\pi}_1$ which are essentially the norms of the fibres of π_2.

As above, if the $\boldsymbol{E}_i$ are normed bundles, we can consider the Banach spaces of *linear* sections which we will denote $\Gamma_{\mathbf{L}}(\bar{\pi}_1)$. Again the space is isometrically isomorphic to the Banach space of vector bundle maps from $\boldsymbol{E}_1$ to $\boldsymbol{E}_2$ whose base maps are the identity. We denote this latter space by $\mathbf{L}_{B,Id}(\boldsymbol{E}_1, \boldsymbol{E}_2)$. In this case the norm is the sup norm of the operator norms,

$$|\sigma| = \sup_{x \in B_1} \left\| s\big|_{\pi_1^{-1}(x)} \right\|.$$

3.4 Bundles of vector bundle maps, $\mathbf{L}(\boldsymbol{E},\boldsymbol{F})$

Consider a pair of normed vector bundles, $\pi_{\boldsymbol{E}} : \boldsymbol{E} \to B_E$ and $\pi_{\boldsymbol{F}} : \boldsymbol{F} \to B_F$, with typical fibres E and F, for which there exists a diffeomorphism (homeomorphism) h between B_E and B_F. We can construct the vector bundle, $\pi : \mathbf{L}_h(\boldsymbol{E},\boldsymbol{F}) \to B_E$, whose fibres consist of the space of linear maps between the respective fibres of the constituent bundles. That is for $x \in B_E$, we have $\pi^{-1}(x) = \mathbf{L}\left(\boldsymbol{E}_x, \boldsymbol{F}_{h(x)}\right)$. This construction is used in two distinct ways in this thesis. In the first use, this construction, with $B_E = B_h$ and $h = Id$ forms the essential step in the proof, in part II, that the Stable manifolds are C^r whenever f is C^r. In its second use, this construction, with $B_E = B_F = M$ and f a given diffeomorphism of the manifold M, is used to prove Hölder inequalities involving the derivative, Df, of f. In particular, Df is a section of the bundle, $\mathbf{L}_f(TM, TM)$.

We claim that, $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F})$ is a normed vector bundle. Since the base space of $\mathbf{L}_f(\boldsymbol{E},\boldsymbol{F})$ is defined to be B_E, we similarly define the base metric of $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F})$ to be the base metric of $\boldsymbol{E}$. We take the fibre norm of $\mathbf{L}(\boldsymbol{E},\boldsymbol{F})$ to be the usual operator norm of the appropriate space of linear maps, $\mathbf{L}\left(\boldsymbol{E}_x, \boldsymbol{F}_{h(x)}\right)$.

This allows us to conclude that $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F})$ is a normed vector bundle. Unfortunately, since the fibre norm is defined via the supremum, the fibre norm will not, in general, be C^r. This in turn means that $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F})$ need not be a C^r normed vector bundle even if both $\boldsymbol{E}$ and $\boldsymbol{F}$ are C^r normed vector bundles. The bundle $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F})$ will, however, always be a continuous normed vector bundle.

3.5 Bundles of bi-linear vector bundle maps, $\mathbf{L}(\boldsymbol{E},\boldsymbol{E};\mathbf{R})$

The Riemannian tensor, $\langle\cdot,\cdot\rangle_\cdot$, is a section of the bundle, $\mathbf{T}^0_2(TM)$. This bundle is, in our case, more easily understood as the bundle of *bi*-linear maps from $TM \oplus_M TM$ into the reals, that is as the bundle, $\mathbf{L}(TM,TM;\mathbf{R})$. In order to view $\mathbf{L}(TM,TM;\mathbf{R})$ as a normed bundle, it is most natural to view $\mathbf{R}$ as the product vector bundle, $\pi : M \times \mathbf{R} \to M$, where $\pi(x,r) = x$ for all $(x,r) \in M \times \mathbf{R}$. When viewed this way, the vector bundle, $\mathbf{R}$, is trivially a normed bundle, where the norm is the standard norm and the function, τ, is trivially the identity.

More generally we could consider the bundle, $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F};\boldsymbol{G})$, of bi-linear bundle maps from $\boldsymbol{E} \oplus_B \boldsymbol{F}$ to $\boldsymbol{G}$ over the base transformation, h. Similar arguments to the ones used in the previous section would show that this bundle, $\mathbf{L}_h(\boldsymbol{E},\boldsymbol{F};\boldsymbol{G})$, is a normed vector bundle so long as $\boldsymbol{E}$, $\boldsymbol{F}$, and $\boldsymbol{G}$ are each normed bundles.

Since we will only need this construction for the specific case, $\mathbf{L}(TM,TM;\mathbf{R})$, we will not deal with the more general case in this thesis. Moreover, since the arguments are really very similar to those used in the last section, we will only state the main definitions and inequalities.

Given a normed vector bundle structure on TM as well as the trivial normed vector bundle structure on $\mathbf{R}$, we define the normed vector bundle structure on $\mathbf{L}(TM,TM;\mathbf{R})$ as follows. For each $x \in M$, the fibre is $\mathbf{L}(T_xM,T_xM;\mathbf{R}_x)$. In each such fibre we define the fibre norm to be the standard bi-linear operator norm

$$|L|_x = \sup_{v,w\in T_xM \,|\, |v|_x = |w|_x = 1} L(v,w),$$

we define the base metric to be the Riemannian distance metric of M.

3.6 Projected Subbundles

When studying uniform hyperbolicity of an invariant set, Λ, one of our main assumptions is that there exists a splitting $T_\Lambda M = \boldsymbol{E}_u \oplus_\Lambda \boldsymbol{E}_s$ of the tangent bundle of Λ. Implicit in the definition of uniform hyperbolicity is the statement that there exists a pair of *globally defined* projections, $p_u : T_\Lambda M \to \boldsymbol{E}_u$, and $p_s : T_\Lambda M \to \boldsymbol{E}_s$ for which the norms $||p_u||_x$ and $||p_s||_x$ are uniformly bounded over the set Λ. This uniform boundedness ensures that, in each fibre, the subspaces $(\boldsymbol{E}_u)_x$ and $(\boldsymbol{E}_s)_x$ are not too "close" in the x-fiber of the Grassmannian bundle formed of the tangent bundle of and invariant set Λ (i.e. $G(T_\Lambda M)_x$).

If a subbundle $\boldsymbol{E}_0$ of a normed vector bundle $\boldsymbol{E}$ is equipped with a globally defined projection, p_0, then the subbundle, $\boldsymbol{E}_0$, can be given a normed vector bundle structure which is naturally inherited from the normed vector bundle structure of $\boldsymbol{E}$ and the projection p_0. We can do this by defining the fibre norm of the subbundle, $|\cdot|_{(0,\cdot)}$, by

$$|e_0|_{(0,x)} = |e_0|_x ,$$

for all $x, y \in B$, and $e_0 \in \boldsymbol{E}_0$.

Chapter 4

Grassmannian manifolds and splittings of TM

The fibre bundle version of the perturbed Stable Manifold theorem proven in Part II *depends* on the existence of a global splitting of the bundle which is invariant under the action of a "hyperbolic" vector bundle morphism. In Part III, in order to show the existence of such a C^β invariant splitting we will consider "pseudo-orbits" of C^0 splittings, and their C^1 approximations. The most natural way in which to consider these splittings is as a C^r map defined on a given Riemannian manifold M, which maps into the Grassmannian manifold, GM, of M.

Recall that the Grassmannian manifold of an n-dimensional manifold M is the set of $0 \leq k \leq n$ dimensional planes contained in the typical fibre of TM. In order to discuss the C^β continuity properties of a given splitting we will need to make the Grassmannian into a metric space. In our case, the most natural metric to give the Grassmannian is one which depends on the norms in the fibres of TM. Since, for a Riemannian manifold, this norm varies from fibre to fibre, the most natural structure in which to discuss these splittings is as a *bundle* over M in which each fibre is the Grassmannian manifold formed from the corresponding fibre of TM. Since this bundle is our primary object, we will denote it by the symbol, GM and call it the Grassmannian bundle of M.

It is important to note that this object is not what is usually called the "Grassmannian bundle". Hirsch, in his book [Hir76], defines the "Grassmannian bundle" to be the "universal bundle over the Grassmannian manifold of M. We will *not* make any use of this "universal bundle" and so no confusion should result from our terminology. See Hirsch's book [Hir76] for a description of this universal bundle.

In fact, the *most* natural object with which to study these splittings is what we will call the double Grassmannian bundle. This double Grassmannian bundle is the subbundle of the cross product of GM with itself whose fibres consist of the cross products of the set of k and $n-k$ planes of the fibres of TM. A splitting of TM is then a special global section of this double Grassmannian bundle.

4.1 Grassmannian manifolds

We begin by defining the Grassmannian manifold for a vector space. While we are only interested in the Grassmannian manifolds of finite dimensional vector spaces, it will be useful for later work, to give a definition of the Grassmannian manifold for any Banach space. See [AMR83] for the general (infinite dimensional) Banach case or [Hir76] for the finite dimensional case. Our definition has been taken from [AMR83].

Fix a Banach space, $\boldsymbol{E}$, and recall that a subspace $\boldsymbol{F}$ of $\boldsymbol{E}$ splits if it is closed and if there exists another closed subspace $\boldsymbol{G}$ of $\boldsymbol{E}$ for which $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$. The *Grassmannian manifold of* $\boldsymbol{E}$, is the set $G(\boldsymbol{E})$ of all split subspaces of $\boldsymbol{E}$, together with the following C^∞ atlas.

For $\boldsymbol{F} \in G(\boldsymbol{E})$ let $\boldsymbol{G}$ denote one of its complements, that is $\boldsymbol{G}$ is a closed subspace of $\boldsymbol{E}$ for which $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$. Define the $\boldsymbol{G}$-neighbourhood of $\boldsymbol{F}$ in $G(\boldsymbol{E})$ to be

$$U_{\boldsymbol{G}} = \{\boldsymbol{H} \in G(\boldsymbol{E}) \mid \boldsymbol{E} = \boldsymbol{H} \oplus \boldsymbol{G}\}.$$

Let $p_{\boldsymbol{F}} : \boldsymbol{E} \to \boldsymbol{F}$ and $p_{\boldsymbol{G}} : \boldsymbol{E} \to \boldsymbol{G}$ denote the canonical projections of $\boldsymbol{E}$ onto $\boldsymbol{F}$ and $\boldsymbol{G}$ *along* $\boldsymbol{G}$ and $\boldsymbol{F}$ respectively. Define the chart map, $\phi_{\boldsymbol{FG}}$, to be

$$\begin{aligned} \phi_{\boldsymbol{FG}} &: U_{\boldsymbol{G}} \to \mathrm{L}(\boldsymbol{F},\boldsymbol{G}), \quad \text{defined by,} \\ \phi_{\boldsymbol{FG}} &= p_{\boldsymbol{HG}} \circ p_{\boldsymbol{HF}}^{-1}, \end{aligned}$$

where $p_{\boldsymbol{HG}} = p_{\boldsymbol{G}}\big|_{\boldsymbol{H}}$ and $p_{\boldsymbol{HF}} = p_{\boldsymbol{F}}\big|_{\boldsymbol{H}}$.

Using Banach's isomorphism theorem, it is easy to show that $H \in U_G$ iff $\phi_{HF} : H \to F$ is an isomorphism, and hence ϕ_{FG} is well defined. In fact, it is similarly easy to show that $H \in U_G$ iff there is a continuous linear map from F into G whose graph in $F \times G \simeq F \oplus G$ is H. This implies that ϕ_{FG} is a bijection.

To show that the set of all such (U_G, ϕ_{FG}) charts forms a C^∞ atlas, consider two points F and $\bar{F}$ in $G(E)$ for which $U_G \bigcap U_{\bar{G}} \neq \emptyset$. Consider $H \in U_G \bigcap U_{\bar{G}}$. By definition we know that $E = H \oplus G = H \oplus \bar{G}$. This implies that G and $\bar{G}$ and hence F and $\bar{F}$ are isomorphic. These isomorphisms define an isomorphism $T : E \to E$ for which $T(F) = \bar{F}$ and $T(G) = \bar{G}$. It is then easy to show that the overlap map,

$$\phi_{FG} \circ \phi_{\bar{F}\bar{G}}^{-1}(\alpha) = \alpha \circ T + (T - Id)$$

is a C^∞ diffeomorphism of $\mathrm{L}\left(\bar{F}, \bar{G}\right)$ to $\mathrm{L}(F, G)$.

Let $G_k(E)$ denote the space of all k-dimensional split subspaces of E. Consider $F \in G_k(E)$ and G such that $E = F \oplus G$. Let $\bar{F} \in G_k(E) \setminus U_G$. This implies that $\bar{F} \subset G$. Let $\bar{G}$ be such that $E = \bar{F} \oplus \bar{G}$. Then (U_G, ϕ_{FG}) and $\left(U_{\bar{G}}, \phi_{\bar{F}\bar{G}}\right)$ are a pair of overlapping charts which together cover $G_k(E)$. This implies that the $G_k(E)$ are connected components of $G(E)$ and so are themselves manifolds.

Our ultimate aim is to work with Grassmannian manifolds of finite dimensional vector spaces, so fix n and consider the vector space, $\mathbf{R}^n$, and fix a given norm, $|\cdot|$. The general definition given above then applies to $\mathbf{R}^n$ and allows us to define the Grassmannian, $G(\mathbf{R}^n)$, of $\mathbf{R}^n$.

In this finite dimensional case it is useful to explicitly add the zero dimensional "manifolds" $G_0(\mathbf{R}^n)$ and $G_n(\mathbf{R}^n)$ which correspond to the "splittings" $E = \mathbf{0} \oplus E$ and $E = E \oplus \mathbf{0}$ respectively. Both $G_0(\mathbf{R}^n)$ and $G_n(\mathbf{R}^n)$ are one point sets which are isomorphic to the "zero dimensional vector space" consisting of a single point.

For $0 \leq k \leq n$, the k^{th} component, $G_k(\mathbf{R}^n)$, of $G(\mathbf{R}^n)$ is a compact connected $k(n-k)$ dimensional manifold.

We can use the given norm of $\mathbf{R}^n$ to make $G(\mathbf{R}^n)$ into a metric space by defining the following metric, for $F, \bar{F} \in G(\mathbf{R}^n)$

$d\left(F, \bar{F}\right) =$

$$\begin{cases} \infty & \text{if} \quad \dim(\boldsymbol{F}) \neq \dim\left(\tilde{\boldsymbol{F}}\right), \\ \max\left\{ \sup\limits_{e \in \boldsymbol{F} \mid |e|=1} \inf\limits_{\tilde{e} \in \tilde{\boldsymbol{F}}} |e - \tilde{e}|, \quad \sup\limits_{\tilde{e} \in \tilde{\boldsymbol{F}} \mid |\tilde{e}|=1} \inf\limits_{e \in \boldsymbol{F}} |e - \tilde{e}| \right\} & \text{otherwise.} \end{cases}$$

For each chart (U_G, ϕ_{FG}) of $G(\mathbf{R}^n)$ the canonical metric of $\mathbf{L}(\boldsymbol{F},\boldsymbol{G})$ is the operator norm. It will be useful to show that the push forward of the metric of $G(\mathbf{R}^n)$, defined above, via the chart map ϕ_{FG} is topologically equivalent to the canonical metric of $\mathbf{L}(\boldsymbol{F},\boldsymbol{G})$. The following lemma does just this.

Lemma 4.1 *Let* $\mathbf{R}^n = \boldsymbol{G} \oplus \boldsymbol{H}$ *be a splitting of* $\mathbf{R}^n$, *let* (U_H, ϕ_{GH}) *denote the chart of* $G(\mathbf{R}^n)$ *associated to this splitting, let* $\tilde{d}_G(\cdot,\cdot)$ *denote the push forward of the metric of* $G(\mathbf{R}^n)$, *and let* $d(\cdot,\cdot)$ *denote the canonical metric of* $\mathbf{L}(\boldsymbol{G},\boldsymbol{H})$. *If* $\boldsymbol{F}, \tilde{\boldsymbol{F}} \in U_H \subset G(\mathbf{R}^n)$, *then*

$$\frac{d(L_F, L_{\tilde{F}})}{(1 + |L_{\tilde{F}}|\,|p_G|)(1 + |L_F|)} \leq \tilde{d}_G\left(\boldsymbol{F}, \tilde{\boldsymbol{F}}\right) \leq \|p_G\|\, d(L_F, L_{\tilde{F}}),$$

where $L_F = \phi_{GH}(\boldsymbol{F})$, *and* $L_{\tilde{F}} = \phi_{GH}\left(\tilde{\boldsymbol{F}}\right)$.

Proof: If $\dim(\boldsymbol{F}) \neq \dim\left(\tilde{\boldsymbol{F}}\right)$ then the equivalence is trivial. Assume instead that $\dim(\boldsymbol{F}) = \dim\left(\tilde{\boldsymbol{F}}\right)$. We prove the second half of the topological equivalence first. Consider $e \in \boldsymbol{F}$ for which $|e| = 1$. Let $e_G = p_G(e)$ and $e_H = p_H(e) = L_F e_G = L_F p_G e$. Let $\tilde{e} = (Id + L_{\tilde{F}})\, p_G e$. Then we know that $|e - \tilde{e}| \leq |L_F - L_{\tilde{F}}|\,|p_G|$. The same argument also works for $e \in \tilde{\boldsymbol{F}}$ for which $|e| = 1$. Hence the second half of the equivalence has been proven.

Now consider the first half of the equivalence. Again, consider $e_G \in \boldsymbol{F}$ for which $|e_G| = 1$, let $e = (Id + L_F)\, e_G$, and $\tilde{e} = (Id + L_{\tilde{F}})\, e_G$. By definition, there exists $\hat{e} \in \tilde{\boldsymbol{F}}$ for which $|e - |e|\,\hat{e}| \leq \tilde{d}_G\left(\boldsymbol{F}, \tilde{\boldsymbol{F}}\right) |e|$. Then $(e - \tilde{e}) = (e - |e|\,\hat{e}) + (|e|\,\hat{e} - \tilde{e})$ and so we know that,

$$0 = p_G(e - \tilde{e}) = p_G(e - |e|\,\hat{e}) + p_G(|e|\,\hat{e} - \tilde{e}),$$

and

$$|(L_F - L_{\tilde{F}})\, e_G| = |e - \tilde{e}| = |e - |e|\,\hat{e}| + ||e|\,\hat{e} - \tilde{e}|.$$

Since,

$$||e|\,\hat{e} - \tilde{e}| \leq |L_{\tilde{F}}|\,|p_G|\, \tilde{d}_G\left(\boldsymbol{F}, \tilde{\boldsymbol{F}}\right) |e|,$$

and moreover, $|e| \leq 1 + |L_{\hat{\boldsymbol{F}}}|$, we know that

$$|(L_{\boldsymbol{F}} - L_{\hat{\boldsymbol{F}}})\, e_G| \leq (1 + |L_{\hat{\boldsymbol{F}}}|\, |p_G|)\, (1 + |L_{\boldsymbol{F}}|)\, \hat{d}_G \left(\boldsymbol{F}, \hat{\boldsymbol{F}}\right).$$

∎

4.2 Grassmannian bundles

We now define the *Grassmannian bundle*, GM, of M to be the bundle whose fibres, $G_x M$, are the Grassmannian manifolds of the finite dimensional tangent spaces, $T_x M$, of M at a point $x \in M$. That is, we define GM so that $G_x M = G(T_x M)$. We again stress that what we have defined as the Grassmannian bundle *is not* what, in particular, Hirsch defines as the Grassmannian bundle in his book [Hir76]. Since we will not require the object that Hirsch defines as the Grassmannian bundle, this discrepancy should not be confusing for the contents of *this* thesis.

Note that since M is a Riemannian manifold, each fibre of TM is equipped with a specific norm which is C^∞ between the fibres. This norm induces a metric in each fibre of GM which is also C^∞ between the fibres of GM.

Note that the only properties of M and TM that we have used in defining the Grassmannian bundle GM, are those which TM has *because* it is a normed vector bundle. This implies that we could define the Grassmannian bundle of any normed vector bundle. Given a general normed vector bundle $\boldsymbol{E}$, we will denote the corresponding Grassmannian bundle by $G\boldsymbol{E}$. The Grassmannian bundles we will use, will in fact be essentially the Grassmannian bundle corresponding to a normed vector bundle derived from TM. Alternatively, since GM is a metric bundle and since all of the normed vector bundle constructions apply equally well to metric bundles we can view the Grassmannian bundles we will use as metric bundles which have been constructed from GM.

4.3 The double Grassmannian

Our primary interest in using the Grassmannian is to topologize the space of all splittings of a Banach space or normed vector bundle. To any point, $\boldsymbol{F}$, in the

Grassmannian manifold, $G(\boldsymbol{E})$, of a Banach space $\boldsymbol{E}$, there corresponds at least one splitting of $\boldsymbol{E}$ itself. Unfortunately for us, there is usually more than one such splitting corresponding to each point, $\boldsymbol{F}$, in $G(\boldsymbol{E})$. We are interested in an individual splitting itself, and in particular, for the work in Parts II and III, given a splitting, $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$, we want to distinguish between the splitting, $\boldsymbol{F} \oplus \boldsymbol{G}$ and the splitting $\boldsymbol{G} \oplus \boldsymbol{F}$. The most natural space in which to study these splittings is the double Grassmannian, $G^2(\boldsymbol{E})$.

The *double Grassmannian manifold*, $G^2(\boldsymbol{E})$, for an n-dimensional normed vector space $\boldsymbol{E}$, is defined to be,

$$G^2(\boldsymbol{E}) = \bigsqcup_{k=0}^{n} G_k(\boldsymbol{E}) \times G_{n-k}(\boldsymbol{E}).$$

Clearly, $G^2(\boldsymbol{E}) \subset G(\boldsymbol{E}) \times G(\boldsymbol{E})$ and so $G^2(\boldsymbol{E})$ can be given the subspace metric induced from the product metric of $G(\boldsymbol{E}) \times G(\boldsymbol{E})$.

Since the constituent parts, $G_k(\boldsymbol{E})$, of the double Grassmannian, $G^2(\boldsymbol{E})$, are all manifolds, the double Grassmannian is itself a manifold. In deed, we note that to any splitting, $\boldsymbol{E} = \boldsymbol{F} \oplus G$, there corresponds a pair of complementary charts, $(U_{\boldsymbol{G}}, \phi_{\boldsymbol{F},\boldsymbol{G}})$ and $(U_{\boldsymbol{F}}, \phi_{\boldsymbol{G},\boldsymbol{F}})$, of the Grassmannian manifold $G(\boldsymbol{E})$. Alternatively, we see that to any such splitting there corresponds the chart,

$$(U_{\boldsymbol{F}\oplus\boldsymbol{G}}, \phi_{\boldsymbol{F}\oplus\boldsymbol{G}}), \quad \text{where}$$

$$\begin{aligned} U_{\boldsymbol{F}\oplus\boldsymbol{G}} &= U_{\boldsymbol{G}} \times U_{\boldsymbol{F}}, \quad \text{and} \\ \phi_{\boldsymbol{F}\oplus\boldsymbol{G}} &= \phi_{\boldsymbol{F},\boldsymbol{G}} \times \phi_{\boldsymbol{G},\boldsymbol{F}}. \end{aligned}$$

Now consider a normed vector bundle, $\pi : \boldsymbol{E} \to B$ with an n-dimensional typical fibre, E. As above we can make a fibrewise extension of the above definition of the double to Grassmannian associated to an n-dimensional normed vector space to define the double Grassmannian, $G^2\boldsymbol{E}$, of a normed vector bundle $\boldsymbol{E}$. Let $G\boldsymbol{E}$ denote the Grassmannian bundle associated to $\boldsymbol{E}$, and let $G_k\boldsymbol{E}$ denote the Grassmannian subbundle of k-planes of $\boldsymbol{E}$. Then the *double Grassmannian manifold*, $G^2\boldsymbol{E}$, is defined to be

$$G^2\boldsymbol{E} = \bigsqcup_{k=0}^{n} G_k\boldsymbol{E} \ \dot{\times}_B \ G_{n-k}\boldsymbol{E}.$$

Again, $G^2E \subset GE \times GE$ and so the total space of G^2E can be given the subspace metric induced from the total space product metric of the $GE \times GE$. Since the constituent parts, G_kE, of the double Grassmannian bundle, G^2E, are all metric bundles, the double Grassmannian bundle is itself a metric bundle.

Note that to any C^r splitting, $E = F \oplus_B G$, of E, there corresponds a pair of *global* C^r sections of GE, $x \mapsto F_x$ and $x \mapsto G_x$. Corresponding to this splitting, is a complementary pair of local trivializations of GE defined over the whole base space. The local trivializations are (U_G, ϕ_{FG}) and (U_F, ϕ_{GF}), and are C^r between and C^∞ within the fibres of GE.

Alternatively, to any such C^r splitting, there corresponds a *global* C^r section of G^2E, $x \mapsto F_x \oplus G_x$. Corresponding to this splitting, is a local trivialization of G^2E defined over the whole base space. In this case the local trivialization is

$$(U_{F\oplus_B G}, \phi_{F\oplus_B G}), \quad \text{where}$$

$$\begin{aligned} U_{F\oplus_B G} &= U_G \times_B U_F, \quad \text{and} \\ \phi_{F\oplus_B G} &= \phi_{FG} \times_B \phi_{GF}. \end{aligned}$$

These "local" trivializations are of particular importance to the work in Part III. In chapter 11 we will show that to any pseudo-hyperbolic pseudo-orbit, $\mathfrak{A}$, there exists of a C^0 splitting for which it is a hyperbolic pseudo-orbit. Then in chapter 12 we will show that any pseudo-orbit, $\mathfrak{B}$, close to $\mathfrak{A}$ (in the base space) has a hyperbolic splitting close to the C^0 splitting of $\mathfrak{A}$. In both of these instances we will be working in a local trivialization induced by an appropriate splitting.

4.4 Spaces of non-degenerate Splittings

The double Grassmannian topologizes the space of all possible splittings for a given Banach space or normed vector bundle. However, for the Weak Shadowing Stable Manifold theorem proven in Part III, we need more than this. For the results in Part III, we are interested in asserting that a given splitting is not too degenerate. That is we want to know that the pair of subspaces of E are not in some sense too close to each other. For us the most natural space in which to study these non-degenerate splittings is the space of K-(nondegenerate)-splittings, $S_K(\boldsymbol{E})$, which we will define in this section.

Given a Banach space, $\boldsymbol{E}$, a positive ε, and a splitting, $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$, the splitting is ***$\varepsilon$-degenerate*** if

1. for every unit vector v_F of $\boldsymbol{F}$ there is a unit vector v_G of $\boldsymbol{G}$ for which $|v_F - v_G| \leq \varepsilon$, and
2. for every unit vector v_G of $\boldsymbol{G}$ there is a unit vector v_F of $\boldsymbol{F}$ for which $|v_G - v_F| \leq \varepsilon$.

A splitting is ***ε-non-degenerate*** if it is not ε-degenerate. However this definition is not the most useful definition for the work of Part III. For our purposes the most useful definition of (non)-degeneracy involves the use of the projection operators induced by a given splitting.

Consider a Hilbert or Banach space, $\boldsymbol{E}$. Associated with any splitting, $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$, of $\boldsymbol{E}$, is a pair of projection operators which project the vectors of $\boldsymbol{E}$ *onto* one of the two subspaces of $\boldsymbol{E}$ *along* the other. For a general splitting of a Banach space, the norms of the projection operators are greater than or equal to one. For a Hilbert space the norms of the projections are equal to one iff the subspaces comprising the splitting are perpendicular. We will always assume that the norm of *any* projection is greater than or equal to one.

Given a Banach space, $\boldsymbol{E}$, a constant $K \geq 1$, and a splitting, $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$, let p_F (p_G) denote the projection of $\boldsymbol{E}$ onto the subspace $\boldsymbol{F}$ ($\boldsymbol{G}$) along the subspace $\boldsymbol{G}$ ($\boldsymbol{F}$). The splitting is ***K-degenerate*** if $\min\{|p_F|, |p_G|\} > K$. The splitting is ***K-non-degenerate*** if it is not K-degenerate, that is if $\max\{|p_F|, |p_G|\} \leq K$. We will use the symbol $S_K(\boldsymbol{E})$ to denote the subspace of splittings, i.e., of the double Grassmannian, $G^2(\boldsymbol{E})$, which are all K-non-degenerate. That is we define

$$S_K(\boldsymbol{E}) = \left\{ \boldsymbol{F} \oplus \boldsymbol{G} \in G^2(\boldsymbol{E}) \;\middle|\; \max\{|p_F|, |p_G|\} \leq K \right\}.$$

Given a K-non-degenerate splitting $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$ and a pair of subspaces, $\bar{\boldsymbol{F}}$ and $\bar{\boldsymbol{G}}$, of $\boldsymbol{E}$ which are close to $\boldsymbol{F}$ and $\boldsymbol{G}$ respectively, we would like to estimate the non-degeneracy of the splitting $\boldsymbol{E} = \bar{\boldsymbol{F}} \oplus \bar{\boldsymbol{G}}$. The splitting $\boldsymbol{E} = \bar{\boldsymbol{F}} \oplus \bar{\boldsymbol{G}}$ provides natural charts, $(U_G, \phi_{F,G})$ and $(U_F, \phi_{G,F})$, of the $\boldsymbol{G}$ and $\boldsymbol{F}$-neighbourhoods of $\boldsymbol{F}$ and $\boldsymbol{G}$ respectively. The naturality of this pair of charts suggests that we make the above estimate in the space of linear maps for $\boldsymbol{F}$ to $\boldsymbol{G}$, $\mathbf{L}(\boldsymbol{F},\boldsymbol{G})$. The following lemma provides the estimate we need.

Lemma 4.2 *Consider a K-non-degenerate splitting $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G}$. Consider a pair of subspaces $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{G}}$ which are of the same dimension as $\boldsymbol{F}$ and $\boldsymbol{G}$ respectively. Let L_F and L_G denote the linear maps contained in $\mathbf{L}(\boldsymbol{F},\boldsymbol{G})$ and $\mathbf{L}(\boldsymbol{G},\boldsymbol{F})$ respectively whose graphs are the subspaces $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{G}}$.*

If $\|L_F\|, \|L_G\| \leq \delta$ and $2\delta K < 1$ then the subspaces. $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{G}}$, form a splitting $\boldsymbol{E} = \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}$ and moreover this splitting is $K\left[\frac{1+2\delta K}{1-2\delta K}\right]^2$-non-degenerate.

Proof: While the above estimate seems a bit crude, it does work and so we provide its proof.

To show that the splitting $\boldsymbol{E} = \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}$ is non-degenerate we have to show that norms of the projection operators, $p_{\tilde{F}} : \boldsymbol{E} \to \tilde{\boldsymbol{F}}$ and $p_{\tilde{G}} : \boldsymbol{E} \to \tilde{\boldsymbol{G}}$ which project *along* $\tilde{\boldsymbol{G}}$ and $\tilde{\boldsymbol{F}}$ respectively are both bounded. To see that the pair of subspaces, $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{G}}$, form a splitting of $\boldsymbol{E}$ and moreover to estimate the norms of the appropriate projections, for each vector $e \in \boldsymbol{E}$ we need to find a unique pair of vectors, $(\tilde{f}, \tilde{g})$, in $\tilde{\boldsymbol{F}} \times \boldsymbol{G}$ for which $e = \tilde{f} + \tilde{g}$.

To do this, for each vector $e \in \boldsymbol{E}$, we use a contraction mapping argument. We work in the Banach space $\tilde{\boldsymbol{F}} \times \tilde{\boldsymbol{G}}$ with the (sum) product norm, $\left|(\tilde{f}, \tilde{g})\right| = \left|\tilde{f} + \tilde{g}\right|$ for $\tilde{f} \in \tilde{\boldsymbol{F}}$ and $\tilde{g} \in \tilde{\boldsymbol{G}}$. Given a vector $e \in \boldsymbol{E}$, our contraction mapping is defined by

$$\Phi_e \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} = \begin{matrix} \tilde{f} + (Id + L_F) \circ p_F \circ \left(e - \tilde{f} - \tilde{g}\right) \\ \tilde{g} + (Id + L_G) \circ p_G \circ \left(e - \tilde{f} - \tilde{g}\right) \end{matrix} .$$

To verify that Φ_e is a contraction mapping, consider $\tilde{f}_1, \tilde{f}_2 \in \tilde{\boldsymbol{F}}$ and $\tilde{g}_1, \tilde{g}_2 \in \tilde{\boldsymbol{G}}$. Since $p_F + p_G = Id$, we have

$$\begin{aligned} &\left|\Phi_e\left(\tilde{f}_1, \tilde{g}_1\right) - \Phi_e\left(\tilde{f}_2, \tilde{g}_2\right)\right| \\ &= \left| \begin{matrix} \tilde{f}_1 + (Id + L_F) \circ p_F \circ \left(e - \tilde{f}_1 - \tilde{g}_1\right) \\ + \tilde{g}_1 + (Id + L_F) \circ p_G \circ \left(e - \tilde{f}_1 - \tilde{g}_1\right) \\ - \tilde{f}_2 - (Id + L_F) \circ p_F \circ \left(e - \tilde{f}_2 - \tilde{g}_2\right) \\ - \tilde{g}_2 - (Id + L_F) \circ p_G \circ \left(e - \tilde{f}_2 - \tilde{g}_2\right) \end{matrix} \right| \end{aligned}$$

$$\begin{aligned} &\le \left|L_F p_F + L_G p_G\right| \left|\tilde{f}_2 + \tilde{g}_2 - \tilde{f}_1 - \tilde{g}_1\right| \\ &\le 2\delta K \left|\tilde{f}_2 + \tilde{g}_2 - \tilde{f}_1 - \tilde{g}_1\right|. \end{aligned}$$

In order to interpret the significance of the fixed points of Φ_e, we will need the additional fact that the linear maps $(Id + L_F) : \boldsymbol{F} \to \tilde{\boldsymbol{F}}$ and $(Id + L_G) : \boldsymbol{G} \to \tilde{\boldsymbol{G}}$ are both invertible. Since these of arguments are essentially the same we will only consider $(Id + L_F)$. To see that this linear map is invertible we only need to show that, for all $f \in \boldsymbol{F}$ for which $|f| > 0$ we have $|(Id + L_F) f| > 0$. Since $K \geq 1$, we know that $\delta < 1$. This means that

$$0 < (1 - \delta) |f| \le (1 - |L_F|) |f| \le |(Id + L_F) f|$$

for all $f \in \boldsymbol{F}$ for which $|f| > 0$. Similar arguments can be used to show that both linear maps are invertible.

Since Φ_e is a contraction mapping of a Banach space, it has a unique fixed point, $\left(\tilde{f}_e, \tilde{g}_e\right)$, for each $e \in \boldsymbol{E}$. By construction we have

$$\begin{aligned} 0 &= (Id + L_F) \circ p_F \circ \left(e - \tilde{f}_e - \tilde{g}_e\right), \quad \text{and} \\ 0 &= (Id + L_G) \circ p_G \circ \left(e - \tilde{f}_e - \tilde{g}_e\right). \end{aligned}$$

Since the maps $(Id + L_F) : \boldsymbol{F} \to \tilde{\boldsymbol{F}}$ and $(Id + L_G) : \boldsymbol{G} \to \tilde{\boldsymbol{G}}$ are both invertible, and moreover $p_F + p_G = Id$ this pair of equations imply that $e = \tilde{f}_e + \tilde{g}_e$. Since any other pair of vectors $\left(\tilde{f}_1, \tilde{g}_1\right)$ for which $e = \tilde{f}_1 + \tilde{g}_1$ would also be a fixed point of Φ_e, we see that $\left(\tilde{f}_e, \tilde{g}_e\right)$ is the unique pair of vectors for which $e = \tilde{f}_e + \tilde{g}_e$. Since we can do this for any $e \in \boldsymbol{E}$, this means that the pair of subspaces, $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{G}}$, form a splitting $\boldsymbol{E} = \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}$.

We are now interested in estimating the norms of $\tilde{f}_e$ and $\tilde{g}_e$ for any $e \in \boldsymbol{E}$. We do this in two steps. Since Φ_e is a contraction mapping, we know that $\Phi_e^n(0,0)$ is a Cauchy sequence whose limit is $\left(\tilde{f}_e, \tilde{g}_e\right)$. In the first step we estimate $\left|\left(\tilde{f}_e, \tilde{g}_e\right) - \Phi_e^n(0,0)\right|$ for any n. In the second step we directly estimate the norms of $\tilde{f}_e$ and $\tilde{g}_e$.

Fix $e \in \boldsymbol{E}$ and consider

$$\left|\left(\tilde{f}_e, \tilde{g}_e\right) - \Phi_e^n(0,0)\right|$$

$$
\begin{aligned}
&\leq \sum_{i=n}^{\infty}\left|\Phi_e^{i+1}(0,0)-\Phi_e^{i}(0,0)\right| \\
&\leq \sum_{i=n}^{\infty}(2\delta K)^{i}\left|\Phi_e(0,0)-(0,0)\right| \\
&\leq \sum_{i=n}^{\infty}(2\delta K)^{i}\left|Id+L_F p_F+L_G p_G\right||e| \\
&\leq \left[\frac{1}{1-2\delta K}-\frac{1-(2\delta K)^n}{1-2\delta K}\right](1+2\delta K)|e| \\
&\leq (2\delta K)^n\left[\frac{1+2\delta K}{1-2\delta K}\right]|e|.
\end{aligned}
$$

Now we estimate the norm of $\tilde{f}_e$. To do this we let $\left(\tilde{f}_e^n, \tilde{g}_e^n\right) = \Phi_e^n(0,0)$. Then we have

$$
\begin{aligned}
\left|\tilde{f}_e^{n+1}\right| &\leq \left|\tilde{f}_e^n\right| + (1+\delta)K\left|\left(\tilde{f}_e, \tilde{g}_e\right)-\Phi_e^n(0,0)\right| \\
&\leq \cdots \\
&\leq (1+\delta)K\left[\frac{1+2\delta K}{1-2\delta K}\right]|e|\sum_{i=0}^{n+1}(2\delta K)^i \\
&\leq K\left[\frac{1+2\delta K}{1-2\delta K}\right]^2|e|.
\end{aligned}
$$

This in turn implies that the norm of the projection, $p_{\tilde{F}}$, is bounded by $K\left[\frac{1+2\delta K}{1-2\delta K}\right]^2$. Similar arguments show that the norm of the projection, $p_{\tilde{G}}$, is also bounded by $K\left[\frac{1+2\delta K}{1-2\delta K}\right]^2$. This means that the splitting $\boldsymbol{E} = \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}$ is $K\left[\frac{1+2\delta K}{1-2\delta K}\right]^2$-non-degenerate as stated. ∎

We are now interested in the definition of the "closeness" of a pair of splittings. While we could use the metric, defined above, for the double Grassmannian, $G^2(\boldsymbol{E})$, it is not the most convenient for our purposes. Since all of the splittings which we will consider are K-non-degenerate for some $K \geq 1$, it will be more useful to use the following alternate "metric" of $G^2(\boldsymbol{E})$ which again uses the norms of the canonical projection operators associated to the given splittings. Given the pair of splittings, $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{G} = \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}$, we define,

$$
d_p\left(\boldsymbol{F}\oplus\boldsymbol{G}, \tilde{\boldsymbol{F}}\oplus\tilde{\boldsymbol{G}}\right) = \max\left\{|p_F p_{\tilde{G}}|, |p_G p_{\tilde{F}}|, |p_{\tilde{F}} p_G|, |p_{\tilde{G}} p_F|\right\}.
$$

For $\boldsymbol{F}\oplus\boldsymbol{G}$ and $\tilde{\boldsymbol{F}}\oplus\tilde{\boldsymbol{G}}$ in $S_K(\boldsymbol{E})$, this is a K-metric. That is it satisfies the following K-triangle inequality

$$
d_p\left(\boldsymbol{F}\oplus\boldsymbol{G}, \check{\boldsymbol{F}}\oplus\check{\boldsymbol{G}}\right) \leq K d_p\left(\boldsymbol{F}\oplus\boldsymbol{G}, \tilde{\boldsymbol{F}}\oplus\tilde{\boldsymbol{G}}\right) + K d_p\left(\tilde{\boldsymbol{F}}\oplus\tilde{\boldsymbol{G}}, \check{\boldsymbol{F}}\oplus\check{\boldsymbol{G}}\right).
$$

The following lemma shows that this K-metric is topologically equivalent to the metric of the double Grassmannian, $G^2(\boldsymbol{E})$.

Lemma 4.3 *Fix $K \geq 1$, and consider a pair of splittings, $\boldsymbol{F} \oplus \boldsymbol{G}$, $\tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}} \in S_K(\boldsymbol{E}) \subset G^2(\boldsymbol{E})$. Then*

1. *$d_p\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right) \leq K^2 d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right)$, and*
2. *if $d_p\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right) \leq \delta < 1$ then there exists a constant positive C_δ which depends only on δ for which $d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right) \leq C_\delta d_p\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right)$.*

Proof: We will prove statement 1 first. We only need to show that $|p_F p_{\tilde{G}}| \leq K^2 d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right)$, since the proofs of the similar statements for the other projection operators are essentially the same.

Fix $\tau > 1$ and consider $e \in \boldsymbol{E}$ for which $p_{\tilde{G}}(e) \neq 0$. Let $\tilde{e} = p_{\tilde{G}}(e)$. Since $\tilde{e} \in \tilde{G}$, there exists $\hat{e} \in G$ for which

$$|\tilde{e} - |\tilde{e}|\,\hat{e}| \leq \tau\,|\tilde{e}|\,d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right).$$

This implies that

$$\begin{aligned}
|p_F p_{\tilde{G}}(e)| = |p_F(\tilde{e})| &= |p_F(\tilde{e}) - p_F(|\tilde{e}|\,\hat{e})| \\
&\leq |p_F|\,|\tilde{e} - |\tilde{e}|\,\hat{e}| \\
&\leq \tau K d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right)|e| \\
&\leq \tau K^2 d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right)|e|.
\end{aligned}$$

Since $\tau > 1$ was arbitrary we have

$$|p_F p_{\tilde{G}}(e)| \leq K^2 d\left(\boldsymbol{F} \oplus \boldsymbol{G}, \tilde{\boldsymbol{F}} \oplus \tilde{\boldsymbol{G}}\right)|e|.$$

If $p_{\tilde{G}}(e) = 0$ then the last inequality is trivially satisfied, and so we have proven the statement 1.

We will now prove statement 2. Let $C_\delta = \frac{\delta}{1-\delta}$, then, since $\frac{x}{1-x}$ is concave up over the interval $[0,1)$, we know that $\frac{x}{1-x} \leq C_\delta x$ for all $0 \leq x \leq \delta$.

We will again only show that for every $e \in F$ for which $|e| = 1$ there exists an $\bar{e} \in \bar{F}$ for which

$$|e - \bar{e}| \leq \frac{|p_G p_{\bar{F}}|}{1 - |p_G p_{\bar{F}}|} \leq C_6 |p_G p_{\bar{F}}| .$$

This plus the similar statements for the other pairs of subspaces is enough to verify statement 2.

We begin by showing that the map, $p_{\bar{F}F} = p_F|_{\bar{F}}$ is an isomorphism. Consider $\bar{e} \in \bar{F}$ for which $|\bar{e}| > 0$. Since F and $\bar{F}$ are closed subspace of E, it is enough to show that $|p_F(\bar{e})| > 0$. But

$$|\bar{e}| \leq |p_F(\bar{e})| + |p_G(\bar{e})| \leq |p_F(\bar{e})| + |p_G p_{\bar{F}}| |\bar{e}| ,$$

and so

$$0 < |\bar{e}| (1 - |p_G p_{\bar{F}}|) \leq |p_F(\bar{e})| .$$

Now consider $e \in F$ for which $|e| = 1$. Since $p_{\bar{F}F}$ is an isomorphism, there exists a unique $\bar{e} \in \bar{F}$ such that $p_F(\bar{e}) = e$. Then

$$\begin{aligned} |e - \bar{e}| = |p_G(\bar{e})| &= |p_G p_{\bar{F}}(\bar{e})| \\ &\leq |p_G p_{\bar{F}}| |\bar{e}| \\ &\leq \frac{|p_G p_{\bar{F}}|}{1 - |p_G p_{\bar{F}}|} |e| \\ &\leq C_6 |p_G p_{\bar{F}}| . \end{aligned}$$

■

Chapter 5

Continuity of bundle maps

In this chapter we collect the various continuity results which will be required throughout the rest of the thesis. For general metric spaces we will define the concepts of Lipschitz continuity. For general Banach spaces we consider the Lipschitz properties of the inversion operator which maps the space of invertible linear maps into itself. Finally for bundle maps of metric or normed vector bundles, we define the fibre Lipschitz constant as well as proving a version of the Lipschitz inverse function theorem for normed trivial vector bundles.

5.1 Lipschitz continuity

A map, $f : X \to Y$, between two metric spaces, X and Y, is *Lipschitz continuous* with Lipschitz constant $Lip(f)$, if

$$d_Y(f(x), f(\tilde{x})) \leq Lip(f)\, d_X(x, \tilde{x})$$

for all $x, \tilde{x} \in X$ for which $d_X(x, \tilde{x}) < \infty$.

5.2 Lipschitz continuity for bundle maps

Now consider a pair of (metric or normed) bundles, $\pi_E : \boldsymbol{E} \to B_E$ and $\pi_F : \boldsymbol{F} \to B_F$ and a bundle map, $f : \boldsymbol{E} \to \boldsymbol{F}$ between them. If the vector bundle is trivial, then there are two related natural definitions of "Lipschitz" continuity which can be applied to the bundle map f.

The first definition, is only defined for trivial bundles. For a metric or normed bundle which is trivial, we can define a total space metric by taking the box metric formed of the fibre and base metrics. Consider two points, e and $\bar{e}$ in a trivial bundle $\boldsymbol{E}$ which are in different fibres, $e \in \boldsymbol{E}_x$ and $\bar{e} \in \boldsymbol{E}_{\bar{x}}$. Since the bundle is trivial, there is a unique $\bar{\bar{e}} \in \boldsymbol{E}_{\bar{x}}$ which corresponds to e, similarly there is a unique $\bar{\bar{e}} \in \boldsymbol{E}_x$ which corresponds to $\bar{e}$. We can then define the total space distance between the points e and $\bar{e}$ as

$$\mathfrak{d}(e, \bar{e}) = \max\left\{d(x, \bar{x}), |e - \bar{\bar{e}}|_x, |\bar{e} - \bar{\bar{e}}|_{\bar{x}}\right\}.$$

The first definition of Lipschitz continuity then corresponds to the normal definition of Lipschitz continuity with respect to the total space metrics, $\mathfrak{d}_E(\cdot,\cdot)$ and $\mathfrak{d}_F(\cdot,\cdot)$, of the bundles $\boldsymbol{E}$ and $\boldsymbol{F}$. That is, a (metric or normed) trivial vector bundle map, f, is *Lipschitz continuous* if f is Lipschitz with respect to the total metrics of $\boldsymbol{E}$ and $\boldsymbol{F}$. The related Lipschitz constant is denoted $Lip(f)$.

The second definition of "Lipschitz" continuity, which is valid for any metric of normed bundle, corresponds to the normal definition of Lipschitz continuity of the map, $\Gamma(f)$, between the spaces of sections, $\Gamma(\boldsymbol{E})$ and $\Gamma(\boldsymbol{F})$. A metric or vector bundle map, f, between normed vector bundles is ***fibre Lipschitz continuous*** (Lipschitz continuous in the fibres) if there exists a positive constant, $Lip_f(f)$, for which

$$|f(e) - f(\bar{e})|_{f_0(x)} \leq Lip_f(f)\,|e - \bar{e}|_x$$

for all $x \in B_E$ and $e, \bar{e} \in E_x$. In this case, the constant, $Lip_f(f)$, is called the fibre Lipschitz constant (the Lipschitz constant in the fibres). The similar definition for a metric bundle map between metric bundles will be left to the reader.

If the base map, f_0, is a diffeomorphism (homeomorphism) then, f is fibre Lipschitz continuous iff $\Gamma(f)$ is Lipschitz continuous with respect to the appropriate sup metrics of $\Gamma(\boldsymbol{E})$ and $\Gamma(\boldsymbol{F})$. If f is a vector bundle map between the normed bundles, $\boldsymbol{E}$, and $\boldsymbol{F}$, then, since, $\Gamma(f)$, is a linear map between the Banach spaces $\Gamma(\boldsymbol{E})$ and $\Gamma(\boldsymbol{F})$, $\Gamma(f)$ is Lipschitz continuous iff it is a bounded linear mapping of $\Gamma(\boldsymbol{E})$ into $\Gamma(\boldsymbol{F})$, and moreover the Lipschitz constant is then the operator norm of $\Gamma(f)$.

It is important for most the work in this thesis that, while a Lipschitz bundle map is obviously *fibre* Lipschitz, it's *fibre* Lipschitz constant can be, and often is, distinctly smaller than its *Lipschitz* constant. Most of the bundle maps we will consider will be Lipschitz with a Lipschitz constant of at least 1 since the base map is usually expansive. However, we will usually construct the bundle map so that it contracts the fibres. That is we will construct the bundle map so that its *fibre* Lipschitz constant is strictly *less* than 1.

5.3 Lipschitz bundle maps of the Whitney sum

Some care must be taken when considering the Lipschitz properties of a bundle map which involves the Whitney sum of two bundles, $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. Since the Whitney sum of a pair normed (or metric) bundles can be given any one of three different normed (or metric) bundle structures, we must be careful when dealing with a bundle map of π which is also a bundle map of either $\bar{\pi}_1$ or $\bar{\pi}_2$.

Since the fibres differ between the three bundle structures, Lipschitz continuity and the Lipschitz constants *in the fibres* may differ between the three bundle structures. When considering the map as a map of the fibre bundle π then the appropriate fibre norm is the max norm, but when considering the same map as a map of the fibre bundle $\bar{\pi}_1$ or $\bar{\pi}_2$ then the appropriate fibre norm is essentially the fibre norm of $\boldsymbol{E}_2$ or $\boldsymbol{E}_1$ respectively. Hence, the Lipschitz constant *in the fibres* for a fibre bundle map "of" $\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$ depends crucially on the context.

5.4 Tangent sections

Twice in Part II, once in the C^r-Section theorem and once in the perturbed Stable Manifold theorem, we will consider the Whitney sum of a pair of normed bundles, and we will need to show that a pair of local sections of the bundle $\bar{\pi}_1$ are tangent. We can do this if Given a bundle map, $f : \boldsymbol{E}_1 \to \boldsymbol{E}_2$, between a pair of normed bundles, let $\hat{x} = \pi_1(\hat{e}_1)$ and define the *local fibre Lipschitz constant at* $\hat{e}_1$ to be

$$Lip_{f,\hat{e}_1}(f) = \varlimsup_{\epsilon \to 0} \sup_{e_1 \in B_\epsilon(\hat{e}_1) \bigcap \pi_1^{-1}(\hat{x})} \frac{|f(e_1) - f(\hat{e}_1)|_{f_0(\hat{x})}}{|e_1 - \hat{e}_1|_{\hat{x}}}.$$

Again, the similar definitions for metric bundles will be left to the reader.

Recall that the fibre component, s, of a local section, σ, of the bundle $\bar{\pi}_1 : E_1 \oplus_H E_2 \to E_1$ is a metric bundle map from E_1 to E_2. For the point $\tilde{e}_1 \in E_1$, we define the *slope of σ at $\tilde{e}_1$* to be the local *fibre* Lipschitz constant of the fibre component, s, at $\tilde{e}_1$. Moreover, we say that the pair, σ and $\tilde{\sigma}$, of (local) sections of the bundle, $\bar{\pi}_1$, are *tangent at $\tilde{e}_1$* if the (local) section $\sigma - \bar{\sigma}$ has a zero slope at $\tilde{e}_1$. This means that $Lip_{f,\tilde{e}_1}(s - \bar{s}) = 0$.

Note that the slope of σ is not defined by the local fibre Lipschitz constant of σ, since this would depend on the Lipschitz continuity of σ *between* fibres of E_1. In each of the cases in which the slope of σ will be used, for example in the C^r-Section theorem in Part II, we will be unable to produce bounds on the Lipschitz properties of σ *between* the fibres of E_1. In particular, examples due to Anosov, show that σ need only be Hölder continuous between the fibres of E_1.

5.5 The Lipschitz properties of the inversion map, Inv

The proof of the C^r-Section theorem, in Part II, will require the following elementary Lipschitz properties of the inversion map, $\mathrm{Inv} : \mathbf{GL}(E, E) \to \mathbf{GL}(E, E)$. We will state these properties for a given Banach space E and extend them fibrewise as they are needed.

Lemma 5.1 *Let E be a Banach space, and let* $\mathrm{Inv} : \mathbf{GL}(E, E) \to \mathbf{GL}(E, E)$ *be defined as* $\mathrm{Inv}(A) = A^{-1}$. *Then* Inv *is smooth and*

$$(D^s\mathrm{Inv})_A(B_1, \ldots, B_s) = (-1)^s \sum_{\rho \in S_s} A^{-1} B_{\rho(1)} A^{-1} \cdots B_{\rho(s)} A^{-1} \tag{5.1}$$

where S_s is the symmetric group on s symbols and $B_1, \ldots, B_s$ are linear operators on E.

Let $\boldsymbol{A}(\mu) = \{A \in \mathrm{GL}(E, E) \mid |A^{-1}| \le \mu\}$. Then D^sInv is both uniformly bounded and uniformly Lipschitz on $\boldsymbol{A}(\mu)$.

Proof: We prove equation 5.1 via induction. To start with we have $(D\mathrm{Inv})_A(B) = -A^{-1}BA^{-1}$ (see Abraham *et.al.* [AMR83]). Now assume that equation 5.1 is true for $s-1$. By the Leibnitz' Rule we have

$$D(D^s\mathrm{Inv})_A(B_1,\ldots,B_s) =$$
$$(-1)^s \sum_{\rho\in S_s} \left| \begin{array}{l} (-1)A^{-1}B_{s+1}A^{-1}B_{\rho(1)}A^{-1}\cdots B_{\rho(s)}A^{-1} \\ +(-1)A^{-1}B_{\rho(1)}A^{-1}B_{s+1}A^{-1}\cdots B_{\rho(s)}A^{-1} \\ +\cdots+(-1)A^{-1}B_{\rho(1)}A^{-1}\cdots B_{\rho(s)}A^{-1}B_{s+1}A^{-1} \end{array} \right| .$$

By collecting terms, this implies that equation 5.1 is valid for s.

To show that $D^s\mathrm{Inv}$ is uniformly bounded on the set $\mathcal{A}(\mu)$ we note that equation 5.1 implies that

$$|(D^s\mathrm{Inv})_A(B_1\cdots B_s)| \leq s! \left|A^{-1}\right|^{s+1} |B_1|\cdots|B_s| .$$

Hence the operator norm of $(D^s\mathrm{Inv})_A$ as an element of $\mathrm{GL}^s(\mathrm{GL}(E,E);\mathrm{GL}(E,E))$ is $|(D^s\mathrm{Inv})_A| \leq s!\,\mu^{s+1}$ for all $A \in \mathcal{A}(\mu)$.

To show that $D^s\mathrm{Inv}$ is uniformly Lipschitz on $\mathcal{A}(\mu)$ we use a telescoping sum to note that

$$\begin{aligned} & A^{-1}B_{\rho(1)}A^{-1}\cdots B_{\rho(s)}A^{-1} - \tilde{A}^{-1}B_{\rho(1)}\tilde{A}^{-1}\cdots B_{\rho(s)}\tilde{A}^{-1} \\ = & \left(A^{-1}-\tilde{A}^{-1}\right)B_{\rho(1)}A^{-1}\cdots B_{\rho(s)}A^{-1} \\ & + \tilde{A}^{-1}B_{\rho(1)}\left(A^{-1}-\tilde{A}^{-1}\right)\cdots B_{\rho(s)}A^{-1} \\ & + \cdots + \tilde{A}^{-1}B_{\rho(1)}\tilde{A}^{-1}\cdots B_{\rho(s)}\left(A^{-1}-\tilde{A}^{-1}\right). \end{aligned}$$

Using this result, and the estimate that

$$\left|A^{-1}-\tilde{A}^{-1}\right| = \left|A^{-1}\tilde{A}\tilde{A}^{-1} - A^{-1}A\tilde{A}^{-1}\right| \leq \mu^2\left|A-\tilde{A}\right|$$

we find that equation 5.1 implies that $Lip\left(D^s\mathrm{Inv}\big|_{\mathcal{A}(\mu)}\right) \leq (s+1)!\,\mu^{s+2}$. ∎

5.6 The Lipschitz inverse function theorem for bundle maps

The Lipschitz part of the Unstable Manifold Theorem will require the following two theorems about fibre Lipschitz bundle maps. The first theorem is a version of the Lipschitz inverse function theorem for normed trivial vector bundles.

Given a normed bundle we can always forget the normed structure (the given norm on each fibre), as well as the vector space structure of each fibre. By doing this we end up with a metric bundle.

A vector bundle map is by definition a map from one vector bundle to another which preserves fibres and is a vector space morphism (i.e. a linear map) on each fibre. If the bundle maps used in the next two lemmas were *vector bundle* maps the lemmas would be trivial. It is very important that the bundle maps we are using need not be vector bundle maps.

Theorem 5.2 *Consider the following diagram of metric bundle maps*

$$\begin{tikzcd} \boldsymbol{U} \arrow[r, shift left, "f"] \arrow[r, shift right, "h"'] \arrow[d, dash] & \boldsymbol{V} \arrow[d, dash] \\ E_0 \arrow[r, "f_0"] & F_0 \end{tikzcd}$$

where f is a homeomorphism with Lipschitz inverse, h is a continuous Lipschitz map, $\boldsymbol{U}$ is an open subset of the normed trivial vector bundle $\boldsymbol{E}$, and $\boldsymbol{V} = f(\boldsymbol{U})$ is an open subset of the normed trivial vector bundle $\boldsymbol{F}$. Both $\boldsymbol{U}$ and $\boldsymbol{V}$ are given the Metric bundle structure induced by the normed bundle structures of $\boldsymbol{E}$ and $\boldsymbol{F}$ respectively. The base map, f_0, of both f and h, is a homeomorphism between the topological spaces E_0 and F_0. Let $g = f + h$.

If $Lip_f(h)\, Lip_f(f^{-1}) < 1$, then g is a metric bundle homeomorphism of $\boldsymbol{U}$ onto an open subset of $\boldsymbol{F}$, with Lipschitz inverse whose fibre Lipschitz constant satisfies

$$Lip_f\left(g^{-1}\right) \leq \frac{1}{1/Lip_f(f^{-1}) - Lip_f(f-g)}.$$

Proof: Since g is the sum of two continuous metric bundle maps, it is itself continuous. Hence in order to show that g is a homeomorphism we must show that it is injective and that its inverse is continuous or equivalently that g is open. The proofs that g is injective and open are based on Shub's Lemmas I.1 and I.2 respectively, see [Shu87].

We begin by showing that g is injective. Since $Lip_f(h) = Lip_f(f-g) < \frac{1}{Lip_f(f^{-1})}$ we have for x in $\pi_E(\boldsymbol{U})$ and $e, \bar{e} \in \boldsymbol{U}_x = \pi_E^{-1}(x) \bigcap \boldsymbol{U}$

$$\begin{aligned} |g(e) - g(\bar{e})|_x &\geq |f(e) - f(\bar{e})|_{f_0(x)} - |(g-f)e - (g-f)\bar{e}|_{f_0(x)} \\ &\geq \left[\frac{1}{Lip_f(f^{-1})} - Lip_f(f-g)\right] |e - \bar{e}|_x > 0 \quad \text{if} \quad e \neq \bar{e}. \end{aligned}$$

Hence g is injective in each fibre. Since the base map of g is f_0 which is itself injective, g is injective as a map from $\boldsymbol{U}$ to $g(\boldsymbol{U})$. Moreover, the inverse of g is fibre Lipschitz with a fibre Lipschitz constant, $Lip_f(g^{-1})$, satisfying the condition stated above.

We must now show that g is open. Since f^{-1} is a homeomorphism it is enough to show that $gf^{-1} = (f+h)f^{-1} = Id + hf^{-1}$ is open. Let $v = hf^{-1}$. By assumption we have

$$\lambda = Lip_f(v) \leq Lip_f(h)\, Lip_f\left(f^{-1}\right) < 1,$$

Moreover, since both h and f^{-1} are Lipschitz, v is also Lipschitz. Let

$$K = Lip(v) \leq Lip(h)\, Lip\left(f^{-1}\right).$$

Since, for small enough $r > 0$, the balls of size r with respect to the total metric of $\boldsymbol{F}$ form a basis of the topology of $\boldsymbol{F}$ it is enough to show that $Id + v$ maps any such ball to an open set. Since both g and f^{-1} are injective, it enough to show that the image via $Id + v$ of any such ball contains another such ball. That is, it is enough to show that for any $e \in \boldsymbol{V}$ and $r > 0$ for which $B_r(e) \subset \boldsymbol{V}$ we have

$$B_s((Id + v)e) \subset (Id + v)(B_r(e))$$

for some function, s, of r.

Fix such an e and r. Let $\bar{e} = (Id + v)e$ and $s = \frac{r}{1+K/(1-K)}$. We seek a local bundle map which is an inverse to $(Id + v)$ and whose base map is Id_{F_0}. Equivalently, we seek a local bundle map, $w : B_s(\bar{e}) \to \boldsymbol{F}$, whose base map is Id_{F_0} and which satisfies

$$(Id + w)(B_s(\bar{e})) \subset B_r(e), \quad \text{and} \quad (Id + w)(Id + v) = Id_{\boldsymbol{F}}.$$

Rearranging the last equation we see that we require $w = -v(Id + w)$. This suggests that we should look for w as the fixed point of the operator $\Phi(w) =$

$-v(Id+w)$ defined on the set, Z, of all Lipschitz metric bundle maps from $B_s(\bar{e})$ into F whose base map is Id_{F_0} and for which $w(\bar{e}) = -v(e)$, $Lip_f(w) \le \frac{\lambda}{1-\lambda}$, and $Lip(w) \le \frac{K}{1-K}$. Since the base map of all of the bundle maps in Z is Id_{F_0}, any $w \in Z$ maps the fibres of F identically into themselves. Hence Z is a (complete) Banach space with the sup norm of the fibre norms of F, that is

$$|w| = \sup_{e' \in B_s(\bar{e})} |w(e')|_{\pi_F(e')}.$$

Consider $w \in Z$. We begin by showing that Φ maps Z into itself. Firstly, since the base maps of w, v and Id are all Id_{F_0}, the base map of $\Phi(w)$ is Id_{F_0} also. Secondly, since $w(\bar{e}) = -v(e)$, $(Id+w)\bar{e} = e$ and moreover, $\Phi(w)\bar{e} = -v(Id+w)\bar{e} = -v(e)$. Now, Consider the *fibre* Lipschitz constant of w. We have

$$\begin{aligned}
Lip_f(\Phi(w)) &= Lip_f(-v(Id+w)) \\
&\le Lip_f(v)(Lip_f(Id) + Lip_f(w)) \\
&\le \lambda\left(1 + \frac{\lambda}{1-\lambda}\right) \\
&\le \frac{\lambda}{1-\lambda}.
\end{aligned}$$

Finally, consider the Lipschitz constant of w. As above, we have

$$\begin{aligned}
Lip(\Phi(w)) &= Lip(-v(Id+w)) \\
&\le Lip(v)(Lip(Id) + Lip(w)) \\
&\le K\left(1 + \frac{K}{1-K}\right) \\
&\le \frac{K}{1-K}.
\end{aligned}$$

We claim that Φ is Lipschitz with constant λ in Z, we have

$$\begin{aligned}
|-v(Id+w) + v(Id+w')| &= \sup_{e' \in B_s(\bar{e})} |-v(Id+w)e' + v(Id+w')e'|_{\pi_F(e')} \\
&\le \sup_{e' \in B_s(\bar{e})} \lambda\,|(Id+w)e' - (Id+w')e'|_{\pi_F(e')} \\
&\le \lambda\,|w - w'|.
\end{aligned}$$

Since $\lambda < 1$ and Z is complete we see that there exists a unique fixed point, w, of Φ in Z. This implies that $(Id+w)(Id+v) = Id$.

Since w is an element of Z, it is *Lipschitz* with constant $\frac{K}{1-K}$. Hence, $Id + w$ is Lipschitz with constant $\bar{K} = 1 + \frac{K}{1-K}$ and so

$$\partial_F\left((Id + w)\bar{e}, (Id + w)e'\right) \leq \bar{K}\partial_F(\bar{e}, e').$$

This implies that

$$(Id + w)\left(B_{r/\bar{K}}(\bar{e})\right) \subset B_r(e).$$

Since $r/\bar{K} = s$, the local bundle map, $Id + w$ is the local inverse to $Id + v$ which we required. Since e and r for which $B_r(e) \subset V$ were arbitrary, this in turn implies that $Id + v$ and hence g are open.

We have yet to show that g^{-1} is *Lipschitz*. Since $gf^{-1} = Id + v$, we have that $fg^{-1} = (Id + v)^{-1} = Id + w$, that is $g^{-1} = f^{-1}(Id + w)$. Since f^{-1}, Id and w are Lipschitz, so is g^{-1}. ∎

Lemma 5.3 *Let g be a homeomorphism from an open subset U of the Banach space E onto an open subset V of the Banach space F. If g^{-1} is Lipschitz with constant $Lip(g^{-1}) < \lambda$, then for any $e \in E$ and $r > 0$, for which $B_r(e) \subset U$, we have*

$$\overline{B_{\frac{r}{\lambda}}(g(e))} \subset g\left(\overline{B_r(e)}\right).$$

Proof: Consider $e \in E$ and $r > 0$ for which $B_r(e) \subset U$. Let $g(e) = v$ or equivalently, since g is a homeomorphism, let $g^{-1}(v) = e$. Since $Lip(g^{-1}) < \lambda$, we have

$$\left|g^{-1}(v) - g^{-1}(\bar{v})\right|_E \leq \lambda\,|v - \bar{v}|_F.$$

Since g^{-1} is continuous, we have

$$g^{-1}\left(\overline{B_{\frac{r}{\lambda}}(v)}\right) \subset \overline{B_r(g^{-1}(v))}.$$

Since g is a homeomorphism, by taking g of both sets we have

$$\overline{B_{\frac{r}{\lambda}}(g(e))} \subset g\left(\overline{B_r(e)}\right).$$

∎

Corollary 5.4 *Let g be a metric bundle homeomorphism from an open subset U of the normed bundle $\pi_E : \boldsymbol{E} \to B_E$ onto an open subset V of the normed bundle*

$\pi_F : \boldsymbol{F} \to B_F$. Both $\boldsymbol{U}$ and $\boldsymbol{V}$ have the metric bundle structure induced by the normed structures of $\boldsymbol{E}$ and $\boldsymbol{F}$ respectively.

Consider $x \in B_E$ and $e \in \boldsymbol{U}_x = \pi_E^{-1}(x) \bigcap \boldsymbol{U}$. Let $B_{r,x}(e)$ denote the ball in the fibre over x of radius r about the vector e. Let g_0 denote the base map of g.

If g^{-1} is fibre Lipschitz with constant $Lip_f(g^{-1}) < \lambda$, then for any $x \in B_E$ and any $e \in \boldsymbol{U}_x$, for which $B_{r,x}(e) \subset \boldsymbol{U}_x$, we have

$$\overline{B_{\frac{r}{\lambda},g_0(x)}(g(e))} \subset g\left(\overline{B_{r,x}(e)}\right).$$

Proof: Consider $x \in B_E$. Then $g\big|_{U_x}$, is a homeomorphism from the open set $\boldsymbol{U}_x$ of the fibre $\boldsymbol{E}_x$ to an open subset of the fibre $\boldsymbol{F}_{g_0(x)}$. Since both of these fibres are Banach spaces, we can directly apply the previous lemma. ∎

Part II

The C^r_κ-Section and Unstable Manifold Theorems for κ-slowly varying Fibre bundles

The Weak Shadowing Stable manifold Theorem, to be proven in Part IV, is essentially an application of the fibre bundle version of the Stable manifold theorem proved in this part. By using the Pesin-Mather metric, defined in Part III, we can pass back and forth between a weakly hyperbolic problem on a compact manifold, and a uniformly hyperbolic problem on a paracompact manifold. To be able to make use of this technique, we must prove the fibre bundle version of the Stable Manifold Theorem for fibre bundles with paracompact base spaces.

In passing from the compact to the paracompact setup, uniform bounds on the norms of functions become κ^2-slowly varying bounds on the corresponding functions of the paracompact manifold. Hence we must prove the Stable Manifold Theorem for κ-slowly varying bounds. Fortunately, this is essentially what Pugh and Shub [PS89] have done, for the same reasons, in their proof of Pesin's Stable Manifold Theorem.

The central idea behind Pugh and Shub's Graph transform version of the proof of the Pesin's Stable manifold theorem is really very simple: apply the contraction mapping principle uniformly in each fibre to an appropriately constructed pair of vector bundle morphism and normed vector bundle to conclude that an appropriately chosen space of sections of the bundle is invariant under the action of the bundle morphism. The resulting invariant section *corresponds* to the stable or unstable manifold back on the original manifold. The technique is mostly contained in constructing the correct vector bundle morphism, bundle pair.

To show that the stable and unstable manifolds are C^r is only slightly more difficult. In this case we must make a recursive application of the contraction mapping principle. Moreover at each step in the recursion the invariant section obtained in the previous step is used to construct the next bundle morphism, bundle pair for the current step in the recursion.

In both applications of the contraction mapping principle, the bundle morphism is a fibre contraction. Indeed any continuous fibre contraction has a continuous invariant section. We prove this fact in the first chapter of this part of the thesis. The recursive proof that, under appropriate conditions, a given section is C^r is called the C^r_*-Section theorem. It is also stated and proven in the first chapter of this part of the thesis. Not surprisingly, we can actually show that, again under the appropriate conditions, a given section is $C^{r+\gamma}$. This fact is proven by the $C_*^{r+\beta}$-Section theorem. The interested reader can find the statement and proof of this theorem in Pugh and Shub's paper [PS89]. Finally, the Stable manifold theorem itself is stated and proven in the last chapter of this part of the thesis.

Chapter 6

Fibre contractions and the Section theorems

As part of the statement of the perturbed Stable manifold theorem, in chapter 7, we claim that the sections which represent the stable and unstable manifolds are $C^{r+\beta}$. It is the purpose of the C^r_*-Section and $C^{r+\beta}_*$-Section theorems to provide the proofs of these claims.

It is the purpose of this chapter to define the concepts of Fibre contractions, C^r_* and $C^{r+\beta}_*$ continuity required to state and prove Pugh and Shub's Section Theorems. Our proofs will only slightly generalize those given by Pugh and Shub in section 6 of [PS89].

6.1 Fibre contractions

Consider a pair of bundles, $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, over the same base space, H. Assume that the bundle $\boldsymbol{E}_1$ is either a metric or normed vector bundle and that the bundle $\boldsymbol{E}_2$ is a normed vector bundle. Recall that in chapter 3 we showed how to construct at least two different normed vector bundles out of the Whitney sum, $\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$, of such a pair of bundles. For the definition of a fibre contraction, we consider $\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$ as the bundle, $\pi_1 : \boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2 \to \boldsymbol{E}_1$.

Let $r_1, r_2 : H \to (0, \infty)$, and consider the doubly varying bundle $\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$. A *fibre contraction* of $\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$ into $\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$ over the *fibre* *expansion* of $\triangle_{r_1}\boldsymbol{E}_1$ into $\boldsymbol{E}_1$ is a pair of fibre bundle maps $F : \triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2 \to \boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$, and $b : \triangle_{r_1}\boldsymbol{E}_1 \to \boldsymbol{E}_1$ which makes the diagram

$$\begin{tikzcd}
\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2 \arrow[r, "F"] \arrow[d, dash] & \boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2 \arrow[d, dash] \\
\triangle_{r_1}\boldsymbol{E}_1 \arrow[r, "b"] \arrow[d, dash] & \boldsymbol{E}_1 \arrow[d, dash] \\
H \arrow[r, "h"] & H
\end{tikzcd}$$

commute and satisfies

1. $Lip_f(F) = k < 1$,
2. $F(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2) \bigcap (\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2) \subset \triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$,
3. the bundle map b is invertible, and
4. $\triangle_{r_1}\boldsymbol{E}_1 \subset b(\triangle_{r_1}\boldsymbol{E}_1) \subset \boldsymbol{E}_1$.

We call k the *fibre constant* of F. We define the *base constant* of F to be $\alpha = \frac{1}{Lip_f(b^{-1})}$. The fibre constant measures how sharply F contracts the fibres of $\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$ over $\boldsymbol{E}_1$, and the base constant measures how sharply b^{-1} "contracts" the fibres of the base bundle, $\boldsymbol{E}_1$, and hence how sharply b "expands" the fibres of the base bundle, $\boldsymbol{E}_1$. Note that we do not require that $\alpha > 1$.

Given a fibre contraction F we can define its related *Graph Transform*, $\Gamma_F : \Gamma(\tilde{\pi}_1) \to \Gamma(\tilde{\pi}_1)$, as

$$\Gamma_F(\sigma) = F \circ \sigma \circ b^{-1}.$$

A section of the bundle $\tilde{\pi}_1$ is *F-invariant* if $\Gamma_F(\sigma) = \sigma$. We then have the following all important lemma:

Lemma 6.1 *If F is a fibre contraction, then Γ_F has a unique F-invariant section σ_F. Moreover, since $k < 1$ we know that*

$$|\sigma_F| \leq \frac{1}{1-k}\left|F\left(0_{\triangle_{r_1}\boldsymbol{E}_1}\right)\right|.$$

Proof: Consider $\sigma \in \Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$, since $b^{-1}(\triangle_{r_1}\boldsymbol{E}_1) \subset \triangle_{r_1}\boldsymbol{E}_1$, we know that $\sigma \circ b^{-1}(\triangle_{r_1}\boldsymbol{E}_1)$ is defined and is a subset of $\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$. Moreover, since $F(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2) \bigcap (\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2) \subset \triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$, we know that $\Gamma_F(\sigma) = F \circ \sigma \circ b^{-1} \in \Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$ and so Γ_F maps $\Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$ into itself. Since $Lip_f(F) = k < 1$ we know that for $\sigma, \bar{\sigma} \in \Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$ we have

$$\begin{aligned}
|\Gamma_F(\sigma) - \Gamma_F(\bar{\sigma})| &= \sup_{e_1 \in \triangle_{r_1}\boldsymbol{E}_1}\left|F \circ \sigma \circ b^{-1}(e_1) - F \circ \bar{\sigma} \circ b^{-1}(e_1)\right| \\
&\leq k \sup_{e_1 \in \triangle_{r_1}\boldsymbol{E}_1}\left|\sigma \circ b^{-1}(e_1) - \bar{\sigma} \circ b^{-1}(e_1)\right| \\
&\leq k\,|\sigma - \bar{\sigma}|.
\end{aligned}$$

Hence, Γ_F contracts distances in
$\Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$, and since $\Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$ is complete, this implies that there exists a unique fixed point, σ_F, of Γ_F for which $\Gamma_F(\sigma_F) = \sigma_F$.

This last inequality also implies that $\Gamma_F^n\left(0_{\triangle_{r_1}\boldsymbol{E}_1}\right)$ is a Cauchy sequence in $\Gamma(\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2)$. This allows us to estimate

$$\begin{aligned}
|\sigma_F| &= |\sigma_F - 0| \\
&\leq \sum_{n=0}^{\infty}\left|\Gamma_F^{n+1}(0) - \Gamma_F^{n}(0)\right| \\
&\leq \sum_{n=0}^{\infty} k^n\,|\Gamma_F(0) - 0| \\
&\leq \frac{1}{1-k}\,|\Gamma_F(0)| \\
&\leq \frac{1}{1-k}\sup_{e_1 \in \triangle_{r_1}\boldsymbol{E}_1}\left|F \circ 0 \circ b^{-1}(e_1)\right| \\
&\leq \frac{1}{1-k}\,|F(0)|.
\end{aligned}$$

■

6.2 C^r continuity of an F-invariant section

The Section theorems essentially state that smooth enough fibre contractions, F, have C^r and $C^{r+\beta}$ F-invariant sections. However, since our definitions of "smooth enough fibre contraction" do not include the condition that the base space H be a C^r manifold we must first understand what the above statement means. For $r > 0$, the C^r-Section theorem will require that the bundle, $\boldsymbol{E}_1$, be a normed vector bundle. That is, it will require that $\boldsymbol{E}_1$ be a vector bundle which in turn implies that each fibre of $\boldsymbol{E}_1$ is a finite dimensional Banach space (i.e. vector space). This means that we *can* meaningfully talk about the C^r properties of an F-invariant section *in each fibre of* $\boldsymbol{E}_1$.

The C^r properties of the F-invariant section will be proven by recursively building fibre contractions over the fibre expansion b for which the successive (continuous) invariant sections *are* the sections of tangent planes to the previous section. Since at each step these invariant sections are continuous, the C^r properties *in each fibre of* $\boldsymbol{E}_1$ vary continuously *between* the fibres of $\boldsymbol{E}_1$.

In the course of the proof of the C^r_*-Section theorem, we will require a number of results for C^r functions defined on a Banach space. In particular, we will need a measure of the C^r-size of a function as well as bounds on this size obtained by application of the Higher Order Chain Rule (HOCR) and the Higher Order Leibnitz Rule (HOLR). We will also require the results of Lemma 5.1 given in Part I. We will apply these results to the fibre contraction, F, by extending them $\boldsymbol{E}_1$-fibrewise to the bundle $\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$.

The *C^r-size* of a function, f, of a Banach space, $\boldsymbol{E}$, is $|f|_{C^r} = \max\{|f|_{C^0}, |f|^*_{C^r}\}$, where $|f|_{C^0} = \sup_{e\in\boldsymbol{E}}|f(e)|$ and $|f|^*_{C^r} = \sup\{\|D^i f\| \mid 1 \leq i \leq r\}$.

If f and g are compossible C^r functions of a Banach space, $\boldsymbol{E}$, the HOCR states

$$D^r(f\circ g)_x(v_1,\ldots,v_r) = \sum_\gamma \left(D^t f\right)_{g(x)} (D^\gamma g)_x (v_\gamma)$$

where the multi-index γ ranges over all partitions of $\{1,\ldots,r\}$ into nonempty subsets $\gamma : \gamma_1 \sqcup \cdots \sqcup \gamma_t = \{1,\cdots,r\}$, where v_γ denotes the set of $(v_1,\cdots,v_r)$ arranged into t blocks $(v_{\gamma_1},\cdots,v_{\gamma_t})$ and where v_{γ_i} is the collection of vectors v_j with $j\in\gamma_i$.

The symbol $D^{\gamma}g$ denotes the t-tuple of multilinear maps $\left(D^{|\gamma_1|}g, \cdots, D^{|\gamma_t|}g\right)$. Since higher order derivatives are symmetric the ordering of the v_i and $D^{|\gamma_i|}g$ in the v_{γ_i} and t-tuple (above) do not matter. Since the sum has at most r^r terms we have

$$|f \circ g|_{C^r} \leq r^r \, |f|_{C^r} \, |g|_{C^r} \, . \tag{6.1}$$

If Q_z and R_z are compossible linear operators with a C^r dependence on z, then by the HOLR,

$$D^r \left(Q_z \cdot R_z\right) = \sum_{i=0}^{r} \binom{r}{i} \left(D^{r-i} Q_z\right) \cdot \left(D^i R_z\right).$$

Hence we have

$$|Q_z \cdot R_z|_{C^r} \leq 2^r \, |Q_z|_{C^r} \, |R_z|_{C^r} \, . \tag{6.2}$$

6.3 C^r_κ continuity

A function, $g : H \to (0, \infty)$ is a *κ-slowly varying* function for $\kappa > 1$ if

$$\frac{1}{\kappa} \leq \frac{gh(p)}{g(p)} \leq \kappa$$

for all $p \in H$. We reiterate, that we will often, implicitly, extend such a function, defined on H, to a similar function, with the same symbols, defined on $\boldsymbol{E}_1$ or even $\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2$. Recall that, in such a case, we define the extended function to be the pull back of the original function via the appropriate projection, π_1 or $\pi_{12} = \pi_1 \circ \bar{\pi}_1$.

Let $\mathbf{V}_\kappa$ denote the set of all strictly positive functions defined on H which are κ-slowly varying. Note that $\mathbf{V}_\kappa$ is closed under sums, positive multiples, convex combinations and pointwise convergence. If $g_1, g_2 \in \mathbf{V}_\kappa$ then their product belongs to $\mathbf{V}_\kappa^2 = \mathbf{V}_{\kappa^2}$. The C^r_κ-Section theorem requires growth rates in $\mathbf{V}_\kappa^{\varrho(r)}$ where $\varrho(r) = 3^{3^r}$. The important property of $\varrho(r)$ is

$$(r + 4 + r\varrho(r))\, \varrho(r) \leq \varrho(r+1), \tag{6.3}$$

for $r \geq 0$.

Recall that we can also consider $\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2$ as a bundle, $\pi : \triangle_{r_1}\boldsymbol{E}_1 \oplus_H \triangle_{r_2}\boldsymbol{E}_2 \to H$ over the base space H. A fibre contraction, F, is *C^r_κ continuous* if it is C^r in the fibres of E_1 and moreover its C^r-size in the fibres of E_1 is $\mathbf{V}_\kappa$-bounded. That is, if there exists some $B \in \mathbf{V}_\kappa$ for which for all $p \in H$, $\left| F|_{\pi^{-1}(p)} \right|_{C^r} \leq B(p)$. A section, σ, of $\bar{\pi}_1 : \Delta_{r_1}E_1 \oplus_H \Delta_{r_2}E_2 \to E_1$, is C^r_κ *continuous* if its fibre component, s, is C^r in each fibre of E_1, and if there exists some $\bar{B} \in \mathbf{V}_\kappa$ for which for all $p \in H$, $\left| s|_{\pi_1^{-1}(p)} \right|_{C^r} \leq \bar{B}(p)$.

Finally, for $r \geq 0$, the fibre contraction, F, is an *r-fibre contraction*, if it is C^r_κ, a fiber contraction over the fiber expansion b with fiber constant $k = Lip_f(F)$ and base constant $\alpha = \frac{1}{Lip_f(b^{-1})}$, if $k\alpha^{-s}\kappa^{\theta(r)} < 1$ for all $0 \leq s \leq r$, and if the varying disc bounds, r_1 and r_2, are κ-slowly varying functions of H.

6.4 The C^r_κ-Section Theorem

We can now state the C^r_κ-section theorem due to Pugh and Shub [PS89].

Theorem 6.2 **(C^r_κ-Section Theorem)** *If F is a C^r_κ r-fibre contraction then it has a unique F-invariant section σ_F which is $C^r_{\kappa\theta(r)}$.*

Proof: Throughout the proof we will simplify the notation by defining $D = \Delta_{r_1}E_1 \oplus_H \Delta_{r_2}E_2$ and $E = E_1 \oplus_H E_2$. The proof is a simple induction argument. For the base step, we will show that, if F is a C^0_κ 0-fibre contraction then there is a unique F invariant section σ_F of the $\bar{\pi}_1$ bundle which is $C^0_{\kappa\theta(0)}$ continuous.

It will be convenient to break the induction step into two parts. For a first part of the induction step, consider $r \geq 1$, and let

$$\tilde{\kappa} = \kappa^{r+1+(r-1)\theta(r-1)}.$$

We will then show that, if F is a C^r_κ r-fibre contraction whose unique F-invariant section, σ_F is $C^t_{\kappa\theta(t)}$ for all $0 \leq t \leq r-1$, then there exists a $C^{r-1}_{\tilde{\kappa}}$ $(r-1)$-fibre contraction, which we will denote, $\mathbf{L}F$, whose unique $\mathbf{L}F$-invariant section, denoted, τ_F is the section of tangent planes to the F-invariant section σ_F.

Assume, for the moment that the base step as well as the first part of the induction step have both been proven. The base step, of course, verifies the whole theorem for $r = 0$.

Now to complete the induction step, consider $r \geq 1$ and assume that the whole theorem is true for all s-fibre contractions for all $s < r$. Consider a C^r_κ r-fibre contraction F. Since $\varrho(s) < \varrho(r)$ for all $0 \leq s < r$, we note that, in particular, F is a C^s_κ s-fibre contraction for all $s < r$. The theorem then allows us to conclude that the unique F-invariant section, σ_F, is $C^s_{\kappa^{\varrho(s)}}$ for all $s < r$. The first part of the induction step then implies that there exists a C^{r-1}_κ $(r-1)$-fibre contraction, $\mathbf{L}F$, whose unique $\mathbf{L}F$-invariant section, τ_F, is the section of tangent planes to σ_F. Since $\mathbf{L}F$ is a C^{r-1}_κ $(r-1)$-fibre contraction, we can again apply the theorem to conclude that τ_F is $C^{r-1}_{\kappa^{\varrho(r-1)}}$. That is the C^{r-1}-size of τ_F is $\mathbf{V}_\kappa^{\varrho(r-1)}$ bounded. Inequality 6.3 implies that $\kappa^{\varrho(r-1)} \leq \kappa^{\varrho(r)}$. This in turn implies that the C^r-size of σ_F is $\mathbf{V}_\kappa^{\varrho(r)}$ bounded and hence the theorem is true for all r-fibre contractions.

We must now verify the base and first part of the induction step as stated above. We prove the base step first. We do this as a lemma in its own right in order to stress that, in this case, $\boldsymbol{E}_1$ need only be a metric bundle. We will make use of the metric bundle version of this lemma again in Part III.

Lemma 6.3 **(C^0_κ-Section Lemma)** *If F is C^0_κ, $k\kappa < 1$, the varying disc bounds r_1 and r_2 are κ-slowly varying and $\boldsymbol{E}_1$ is a* metric *bundle, then the unique F-invariant section, σ_F, is $C^0_{\kappa^{\varrho(0)}}$, and*

$$|\sigma_F| \leq \frac{1}{1-k}\left|F\left(0_{\Delta_1, \boldsymbol{E}_1}\right)\right|.$$

If F is a C^0_κ 0-fibre contraction then σ_F is $C^0_{\kappa^{\varrho(0)}}$.

Proof: We stress that $\boldsymbol{E}_1$ is a metric bundle which may or may not also be a normed vector bundle. If $\boldsymbol{E}_1$ is a normed vector bundle, then the metric bundle structure which we use is the metric structure inherited from the normed vector bundle structure of $\boldsymbol{E}_1$.

Since $k < 1$, the proof of lemma 6.1 proves that Γ_F contracts the complete metric space $\Gamma(D)$ into itself. Since F *and* b^{-1} are continuous, Γ_F maps the closed subset of continuous sections into itself. Hence the unique fixed point σ_F of Γ_F is continuous. Since D has $\mathbf{V}_\kappa$-bounded fibres, σ_F is of class C^0_κ. Since $r = 0$, $\kappa^{\varrho(0)} = \kappa^3 \geq \kappa$, and so σ_F is also of class $C^0_{\kappa^{\varrho(r)}}$. ∎

We now proceed to prove the first part of the induction step. Consider $r \geq 1$ and assume that F is an r-fibre contraction whose unique F-invariant section, σ_F, is C^s_u for all $s < r$. Furthermore, assume that E_1 is a normed vector bundle. This part of the induction step will itself consist of three major steps. The first step will construct the associated fibre contraction $\mathbf{L}F$. The second step will show that $\mathbf{L}F$ is an $(r-1)$-fibre contraction, and the final step will show that the unique $\mathbf{L}F$ invariant section, τ_F, is the section of tangent planes to σ_F.

Step 1: Constructing $\mathbf{L}F$ We are interested in constructing a new normed vector bundle over the given bundle $\boldsymbol{E}_1$ along with an associated fibre contraction $\mathbf{L}F$ over the given fibre expansion b. We begin by considering the action of F on the tangent planes to the image of σ_F.

A plane at the point $e \in \boldsymbol{D}$ will be taken by D_eF to a plane at $F(e)$. Let $e = e_1 + e_2 \in \boldsymbol{D}$, and $p = \pi(e) = \pi_1(e_1) = \pi_2(e_2)$. Let $\bar{e} = F(e)$ and $\bar{p} = \pi(\bar{e})$. Finally, let $E_1 = \pi_1^{-1}(p) = (\boldsymbol{E}_1)_p$, $E_2 = \pi_2^{-1}(p) = (\boldsymbol{E}_2)_p$, $\bar{E}_1 = \pi_1^{-1}(\bar{p}) = (\boldsymbol{E}_1)_{\bar{p}}$, and $\bar{E}_2 = \pi_2^{-1}(\bar{p}) = (\boldsymbol{E}_2)_{\bar{p}}$.

Recall that σ_F is a section of $\hat{\pi}_1$, and hence of the form, $\sigma_F(e_1) = e_1 + s_F(e_1)$ where the fibre component, s_F, of σ_F, is a metric bundle map from $\boldsymbol{E}_1$ to $\boldsymbol{E}_2$. This implies that, when considering the action of the derivative of F at e, we are only interested in the planes which are the graphs of linear maps from E_1 into E_2. We write the derivative of F at e as the matrix

$$D_eF = \begin{pmatrix} A_e & B_e \\ C_e & D_e \end{pmatrix},$$

where $A_e \in \mathbf{L}\left(E_1, \bar{E}_1\right)$, $B_e \in \mathbf{L}\left(E_2, \bar{E}_1\right)$, $C_e \in \mathbf{L}\left(E_1, \bar{E}_2\right)$, and $D_e \in \mathbf{L}\left(E_2, \bar{E}_2\right)$.

Given a linear map $P \in \mathbf{L}(E_1, E_2)$ the action of D_eF on the graph of P is

$$D_eF\begin{pmatrix} Id \\ P \end{pmatrix} = \begin{pmatrix} A_e + B_eP \\ C_e + D_eP \end{pmatrix}.$$

Since F preserves the fibres of $\hat{\pi}_1$, $B_e = 0$. Since b is an embedding, A_e is invertible. Hence the natural action of F on $P \in \mathbf{L}(E_1, E_2)$ considered as a plane at the point $e \in \boldsymbol{D}$ is $\mathbf{P}F_e : P \to (C_e + D_eP) \circ A_e^{-1}$.

We can extend the fibrewise action of $\mathbf{P}F_e$ to an *affine* bundle morphism of the following bundle

$$\begin{tikzcd} D \oplus_H \mathrm{L}(E_1, E_2) \arrow[r, "\mathbf{P}F"] \arrow[d, dash] & E \oplus_H \mathrm{L}(E_1, E_2) \arrow[d, dash] \\ D \arrow[r, dash, "F"] & E \end{tikzcd}$$

which moreover commutes. The bundle map $\mathbf{P}F$ is affine in each fibre since, at any $e \in D$, the zero linear map of $\mathrm{L}(E_1, E_2)$ is mapped by $\mathbf{P}F_e$ to the linear map, $C_e A_e^{-1} \in \mathrm{L}\left(\hat{E}_1, \hat{E}_2\right)$. The bundle map $\mathbf{P}F$ is C_*^{r-1} in the fibres of E_1, and, since the minimum expansion of A_e is at least α and $|D_e| < k$, it contracts the fibres of $D \oplus_H \mathrm{L}(E_1, E_2)$ more sharply than $k\alpha^{-1} < 1$, that is $Lip_f(\mathbf{P}F) \leq k\alpha^{-1} < 1$.

Unfortunately this is not yet the bundle that we require. Firstly the base map, F, is *not* a fibre expansion. More importantly in order to show that τ_F is the section of tangent planes to the section σ_F, the new fibre contraction must act *only* on planes over points in the graph of σ_F.

Since C_e is part of $D_e F$ there exists a κ-slowly varying function B such that $|C_e| \leq B(\pi(e))$ for all $e \in D$. Choose a constant c large enough to satisfy the condition $\kappa(1 + ck)\alpha^{-1} < c$, and define

$$\mathbf{L}F(e_1 + P) = (b(e_1) + \mathbf{P}F(\sigma_F(e_1) + P)),$$

and let $L = \triangle_{r_1} E_1 \oplus_H \triangle_{cB} \mathrm{L}(E_1, E_2)$. The fibre contraction we seek is then

$$\begin{tikzcd}
\boldsymbol{L} \arrow[r, "\mathbf{L}F"] \arrow[d, dash] & \boldsymbol{E}_1 \oplus_H \mathbf{L}(\boldsymbol{E}_1, \boldsymbol{E}_2) \arrow[d, dash] \\
\triangle_{r_1}\boldsymbol{E}_1 \arrow[r, "b"] \arrow[d, dash] & \boldsymbol{E}_1 \arrow[d, dash] \\
H \arrow[r, "h"] & H
\end{tikzcd}$$

Step 2: $\mathbf{L}F$ is an $(r-1)$-fibre contraction We must first show that $\mathbf{L}F$ is a *fibre contraction*, then we will show that it is an $(r-1)$-fibre contraction. In order to show that $\mathbf{L}F$ is indeed a fibre contraction we need only consider the first two conditions of being a fibre contraction. Firstly, since $\mathbf{L}F$ is a restriction of $\mathbf{P}F$ to the section σ_F, $Lip_f(\mathbf{L}F) \leq Lip_f(\mathbf{P}F) \leq k\alpha^{-1} < 1$. Secondly by choice of the constant c we have

$$\begin{aligned}
|\mathbf{P}F(\sigma_F(e_1)+P)|_{F\sigma_F(e_1)} &\leq \left\|C_{\sigma_F(e_1)} + D_{\sigma_F(e_1)}P\right\| \left\|A^{-1}_{\sigma_F(e_1)}\right\| \\
&\leq (B(e_1) + kcB(e_1))\,\alpha^{-1} \\
&\leq \kappa(1+ck)\,\alpha^{-1}B(b(e_1)) \\
&\leq cB(b(e_1)),
\end{aligned}$$

for all $e_1 + P \in \boldsymbol{L}$. This implies that $\mathbf{L}F(\boldsymbol{L}) \bigcap (\triangle_{r_1}\boldsymbol{E}_1 \oplus_H \boldsymbol{E}_2) \subset \boldsymbol{L}$.

We are now interested in showing that $\mathbf{L}F$ is in fact an $(r-1)$-fibre contraction. Since $\kappa > 1$, $\kappa < \tilde{\kappa}$ and so the fibres of the disc bundle, $\boldsymbol{L}$, are $\mathbf{V}_{\tilde{\kappa}}$ bounded. Since $\mathbf{L}F$ has a fibre constant of $k\alpha^{-1} < 1$ and a base constant of α, in order to show that $\mathbf{L}F$ is a $C^{r-1}_{\tilde{\kappa}}$ $(r-1)$-$\tilde{\kappa}$-fibre contraction we must show

1. the C^{r-1} size of $\mathbf{L}F$ is $\mathbf{V}_{\tilde{\kappa}}$ bounded, and

2. $(k\alpha^{-1})\,\alpha^{-s}\tilde{\kappa}^{s(r-1)} < 1$ for all $0 \leq s \leq r-1$.

The second condition is an easy consequence of the similar condition on F and inequality 6.3. Proving the first condition is more difficult.

Since

$$\mathbf{L}F = b \oplus \left(\mathbf{P}F \circ \left(\sigma_F \oplus Id_{\mathbf{L}(E_1,E_2)}\right)\right),$$

and b is of class C^r_{κ}, it is enough to find bounds on $|\mathbf{P}F \circ (\sigma_F \oplus Id)|_{C^t}$. Applying the inequality derived from the HOCR, inequality 6.1, we have

$$|\mathbf{P}F \circ (\sigma_F \oplus Id)|_{C^t} \leq t^t \, |\mathbf{P}F|_{C^t} \, |\sigma_F \oplus Id|^t_{C^t}, \qquad \text{for all } \; 1 \leq t \leq r-1.$$

Using the inequality derived from the HOLR, inequality 6.2, we have

$$|\mathbf{P}F|_{C^t} \leq \sup_{e \in D} |\mathbf{P}F_e|_{C^t} \leq \sup_{e \in D} \sup_{P \in \triangle_{cB}\mathbf{L}(E_1,E_2)} 2^t \, |(C_e + D_e P)|_{C^t} \left|A_e^{-1}\right|_{C^t}.$$

Since F is of class C^r_{κ} and $P \in \triangle_{cB}\mathbf{L}(E_1, E_2)$ is $\mathbf{V}_\kappa$ bounded, we see that $|(C_e + D_e P)|_{C^t}$ is $\mathbf{V}^2_\kappa$ bounded. The inequality derived from the HOCR applied to $A_e^{-1} = \mathrm{Inv} \circ (e \rightarrow A_e)$ give us

$$\left|A_e^{-1}\right|_{C^t} \leq t^t \, |\mathrm{Inv}|_{C^t} \, |A_e|^t_{C^t}.$$

Since $\|A_e^{-1}\| \leq \alpha^{-1}$, Lemma 5.1 assures us that the higher derivatives of Inv are also *uniformly* bounded. Hence $|A_e^{-1}|_{C^t}$ is $\mathbf{V}^t_\kappa$ bounded, and so $|\mathbf{P}F|_{C^t}$ is $\mathbf{V}^{2+t}_\kappa$ bounded.

By the induction assumption σ_F, and hence $(\sigma_F \oplus Id)$, are of class $C^t_{\kappa\theta(t)}$ for $0 \leq t \leq r-1$. Hence $|\mathbf{L}F|_{C^t}$ is $\mathbf{V}^{2+t+t\theta(t)}_\kappa$ bounded for $0 \leq t \leq r-1$. Taking $t = r-1$ we can conclude that $\mathbf{L}F$ is a $C^{r-1}_{\bar\kappa}$ $(r-1)$-$\bar\kappa$-fibre contraction and hence there exists a unique $\mathbf{L}F$ invariant section τ_F which is of class $C^{r-1}_{\bar\kappa\theta(r-1)}$.

Step 3: τ_F is the section of tangent planes to σ_F Finally, as the last step, we must show that τ_F is the section of tangent planes to the F invariant section σ_F.

Let s_F and t_F denote the fibre components of σ_F and τ_F respectively. Consider $\xi \in \triangle_{r_1}E_1$ and let $p = \pi_1\left(\xi\right)$. Define

$$\begin{aligned} l_\xi &: (\triangle_{r_1}E_1)_p \rightarrow (\triangle_{r_2}E_2)_p \;\; \text{by} \\ l_\xi(e) &= s_F\left(\xi\right) + t_F\left(\xi\right)(e) \end{aligned}$$

and let $\lambda_{\bar{\xi}}$ denote the local section of $D|_{\pi_1^{-1}(\bar{x})}$ defined by $\lambda_{\bar{\xi}}\left(\bar{\xi}+h\right) = \bar{\xi}+h+l_{\bar{\xi}}(h)$ for all $h \in (\Delta_{r_1}\boldsymbol{E}_1)_p$. By construction, $l_{\bar{\xi}}$ is the fibre component of $\lambda_{\bar{\xi}}$ in $\Delta_{r_1}\boldsymbol{E}_1 \oplus \Delta_r \mathrm{L}(\boldsymbol{E}_1, \boldsymbol{E}_2)$. We will show that for all $\bar{\xi} \in \Delta_{r_1}\boldsymbol{E}_1$, the local sections σ_F and $\lambda_{\bar{\xi}}$ are tangent at $\bar{\xi}$.

Let $\Gamma_{cB}(\boldsymbol{D})$ denote the set of sections of D whose slope at any $\bar{\xi} \in \Delta_{r_1}\boldsymbol{E}_1$ is less than or equal to $cB(\bar{e}_1)$. Since the slope of a section at the point $\bar{\xi}$ is, by definition, the local fibre Lipschitz constant of the section's fibre component at $\bar{\xi}$, $\Gamma_{cB}(\boldsymbol{D})$ is a closed subset of $\Gamma(\boldsymbol{D})$. By the arguments used above, we know that DF maps any plane at the point $e \in \boldsymbol{D}$ with slope less than or equal to $cB(e)$ into a plane at the point $F(e)$ with slope less than or equal to $cB(F(e))$. This implies that F itself maps $\Gamma_{cB}(\boldsymbol{D})$ into itself. Hence the unique fixed point σ_F of Γ_F is contained in $\Gamma_{cB}(\boldsymbol{D})$.

Consider two local sections σ and $\bar{\sigma}$ of $\boldsymbol{D}$ which agree at the point $\bar{\xi}_0 \in \Delta_{r_1}\boldsymbol{E}_1$ both of which have slopes less than or equal to $cB\left(\bar{\xi}\right)$. Then $\Gamma_F(\sigma)$ and $\Gamma_F(\bar{\sigma})$ are local sections of $\boldsymbol{D}$ with slopes less than or equal to $cB\left(b\left(\bar{\xi}\right)\right)$ and which agree at the point $\bar{\xi}_1 = b\left(\bar{\xi}_0\right)$. Let $p_0 = \pi_1\left(\bar{\xi}_0\right)$ and $p_1 = \pi_1\left(\bar{\xi}_1\right) = h(p_0)$. Let s and $\bar{s}$ denote the fibre components of σ and $\bar{\sigma}$ respectively. Let f denote the fibre component of F. Let $\Delta_{\bar{\xi}_0} = \Delta_{\bar{\xi}_0}(\sigma, \bar{\sigma}) = Lip_{\bar{\xi}_0}(s - \bar{s})$, and finally let $\Delta_{\bar{\xi}_1} = \Delta_{\bar{\xi}_1}(\Gamma_F(\sigma), \Gamma_F(\bar{\sigma}))$. Recall that $B_{\epsilon,p_1}\left(\bar{\xi}_1\right)$ denotes the ball *in the fibre* $(\Delta_{r_1}\boldsymbol{E}_1)_{p_1}$ of size ϵ about the point $\bar{\xi}_1$. Then

$$
\begin{aligned}
\Delta_{\bar{\xi}_1} &= \lim_{\epsilon\to 0} \sup_{\xi_1 \in B_{\epsilon,p_1}(\bar{\xi}_1)} \frac{|f\sigma b^{-1}(\xi_1) - f\bar{\sigma}b^{-1}(\xi_1)|_{p_1}}{\left|\xi_1 - \bar{\xi}_1\right|_{p_1}} \\
&\leq \lim_{\epsilon\to 0} \sup_{\xi_0 \in B_{\epsilon,p_0}(\bar{\xi}_0)} k\frac{|s(\xi_0) - \bar{s}(\xi_0)|_{p_0}}{\left|\xi_0 - \bar{\xi}_0\right|_{p_0}} \frac{\left|\xi_0 - \bar{\xi}_0\right|_{p_0}}{\left|\xi_1 - \bar{\xi}_1\right|_{p_1}} \\
&\leq k\Delta_{\bar{\xi}_0}\alpha^{-1},
\end{aligned}
$$

where $\xi_1 = b(\xi_0)$.

For $p \in H$, let $\Delta(p) = \sup\left\{\Delta_{\bar{\xi}}\left(\lambda_{\bar{\xi}}, \sigma_F\right) \mid \bar{\xi} \in (\Delta_{r_1}\boldsymbol{E}_1)_p\right\}$ Since τ_F is $\mathbf{V}_n$ bounded, the slope of $\lambda_{\bar{\xi}}$ is $\mathbf{V}_n$ bounded. This and the fact that the slope of σ_F is $\mathbf{V}_n$ bounded implies that $\Delta(p)$ is a $\mathbf{V}_n$ bounded function.

Since the slope of a local section is by definition the local *fibre* Lipschitz constant of the *fibre* component of the section, we are really considering the local

Lipschitz constant (slope) of a map between two Banach spaces. This means that for any given, $\bar{\xi}_0 \in \Delta_{r_1}E_1$, we can use Taylor's theorem about the point $\bar{\xi}_1 = b\left(\bar{\xi}_0\right)$ to approximate

$$\begin{aligned}
&\Gamma_F\left(\lambda_{\bar{\xi}_0}\right)\left(\bar{\xi}_1 + h\right) \\
&\quad = \Gamma_F\left(\lambda_{\bar{\xi}_0}\right)\left(\bar{\xi}_1\right) + D_{\bar{\xi}_1}\Gamma_F\left(\lambda_{\bar{\xi}_0}\right)(h) + O\left(|h|^2\right) \\
&\quad = \left\{\bar{\xi}_1 + F \circ s_F \circ b^{-1}\left(\bar{\xi}_1\right)\right\} + \left(D_{\sigma(\bar{\xi}_0)}F \circ D_{\bar{\xi}_0}\lambda_{\bar{\xi}_0} \circ D_{\bar{\xi}_1}b^{-1}\right)(h) + O\left(|h|^2\right) \\
&\quad = \Gamma_F(\sigma_F)\left(\bar{\xi}_1\right) + \mathrm{L}F(\tau_F)\left(\bar{\xi}_1\right)(h) + O\left(|h|^2\right) \\
&\quad = \lambda_{\bar{\xi}_1}\left(\bar{\xi}_1 + h\right) + O\left(|h|^2\right).
\end{aligned}$$

This implies that $\Delta_{\bar{\xi}_1}\left(\Gamma_F\left(\lambda_{\bar{\xi}_0}\right), \Gamma_F(\sigma_F)\right) = \Delta_{\bar{\xi}_1}\left(\lambda_{\bar{\xi}_1}, \sigma_F\right)$, and hence $\Delta(h(p)) \leq k\alpha^{-1}\Delta(p)$. If there exists a $p \in H$ for which $\Delta(p) \neq 0$, then $\Delta\left(h^{-n}(p)\right) \geq \left(k\alpha^{-1}\right)^{-n}\Delta(p)$. Since this quantity grows too fast for $\Delta(p)$ to be $\mathbf{V}_s$ bounded, we have that $\Delta(p) \equiv 0$, and hence τ_F is the section of tangent planes to σ_F as required. ∎

Chapter 7

The Unstable Manifold Theorem for vector bundles

The (general) unstable Manifold theorem for an invariant set, Λ, of a uniformly hyperbolic map, f, of a Riemannian manifold, M, depends on the special form of the unstable manifold theorem for the fixed point of a map, F, of a, possibly infinite dimensional, Banach space. The uniform hyperbolicity of the map f allows us to create a related map F of this Banach space to itself whose fixed point corresponds to the invariant set Λ of the original map f. In order to generalize this proof to a non-uniformly hyperbolic map f, the weak shadowing unstable manifold theorem constructs a related Fibre bundle map F of a Fibre bundle to itself whose f-invariant *sections* correspond to the stable and unstable manifolds we seek.

The special unstable manifold theorem for a fixed point, depends on the map F being Lipschitz close to an invertible hyperbolic linear operator T of the Banach space to itself which preserves a given splitting of the Banach space $\boldsymbol{E} = \boldsymbol{E}^s \oplus \boldsymbol{E}^u$. That is, there exists a number $\lambda < 1$ such that

$$\left\| T\big|_{E^s} \right\| < \lambda, \quad \left\| T^{-1}\big|_{E^u} \right\| < \lambda.$$

The natural generalization of this invertible hyperbolic linear operator T, is an invertible fibre contraction T (T^{-1}) over a fibre expansion T_u (T_u^{-1}) for the following diagram of Banach bundles

$$\begin{tikzcd}
 & E^s \oplus_H E^u \arrow[rrr, shift left, "T"] \arrow[dl, "\bar{\pi}_u"'] \arrow[dd, "\pi"] \arrow[dr, "\bar{\pi}_s"] & & & E^s \oplus_H E^u \arrow[lll, shift left, "T^{-1}"] \arrow[dl, "\bar{\pi}_u"'] \arrow[dd, "\pi"] \arrow[dr, "\bar{\pi}_s"] & \\
E^u \arrow[dr, "\pi_u"'] \arrow[rrr, bend right, "T_u"] & & E^s \arrow[dl, "\pi_s"] & E^u \arrow[dr, "\pi_u"'] & & E^s \arrow[dl, "\pi_s"] \arrow[lll, bend right, "T_s^{-1}"'] \\
 & H \arrow[rrr, shift left, "h"] & & & H \arrow[lll, shift left, "h^{-1}"] &
\end{tikzcd}$$

More precisely, if we denote the zero section of the bundle E^u as 0_{E^u}, then T is a fibre contraction over the fibre expansion $T_u = T\big|_{E^u} = \bar{\pi}_u \circ T \circ 0_{E^u}$ with a fibre contraction constant $k = \lambda < 1$ and a base constant $\alpha = 1/\lambda > 1$. Similarly T^{-1} is a fibre contraction over the fibre expansion $T_s^{-1} = T^{-1}\big|_{E^s} = \bar{\pi}_s \circ T^{-1} \circ 0_{E^s}$ where the fibre contraction constant k is again equal to λ and the base constant α is equal to $1/\lambda$. We call T a *(uniformly) hyperbolic linear (double) fibre contraction* with hyperbolicity constant λ.

Recall from the C^r_*-Section theorem, that the function, $\varrho(r)$, is defined to be $\varrho(r) = 3^{3^r}$. Define the positive function, $\bar{\varrho}(r)$, by firstly defining $\bar{\varrho}([r]) = \prod_{k=0}^{[r]} \varrho(k)$ and then defining $\bar{\varrho}(r) = \bar{\varrho}([r])\,\varrho(r)$. Then the Unstable Manifold Theorem for a normed vector bundle states that for any $r \geq 1$, any small enough $\kappa > 1$, and any hyperbolic linear fibre contraction T, as above, then any $C^{r+\beta}_*$ map of some κ-slowly varying disc bundle $\pi : \triangle_r E^s \oplus_H \triangle_r E^u \to H$ which is sufficiently Lipschitz close to T is sufficiently close to being an r-fibre contraction to have a pair of unique $C^{r+\beta}_{*\bar{\varrho}(r+\beta)}$ f-invariant sections of $\bar{\pi}_u$ and $\bar{\pi}_s$ respectively which intersect at a unique invariant section of π.

The key to the proof of the Unstable Manifold Theorem for a normed vector bundle is to see everything as a fibre contraction map. The stable and unstable manifolds to be found by the Unstable Manifold Theorem will be invariant sections of appropriate fibre bundles $\bar{\pi}_u$ and $\bar{\pi}_s$ for appropriate fibre contraction maps related to the given "almost" fibre contraction map f. The map f is "almost" a fibre contraction *because* it is Lipschitz close to an invertible linear hyperbolic fibre map T. That is both T and its inverse T^{-1} are fibre contractions *and* both f and its inverse f^{-1} are Lipschitz close to T and T^{-1} respectively.

The stable and unstable manifolds will be continuous sections of the varying disc bundles $\bar{\pi}_u : \triangle_r E^s \oplus_H \triangle_r E^u \to \triangle_r E^u$ and $\bar{\pi}_s : \triangle_r E^s \oplus_H \triangle_r E^u \to \triangle_r E^s$ which have certain additional Lipschitz properties. Recall that since $E^s \oplus_H E^u$ is a direct sum of the two Banach bundles E^u and E^s, a section σ of the bundle $\bar{\pi}_u$ is expressed as

$$\sigma(e_u) = e_u + s(e_u)$$

where $s \in C^0_{Id}(E^u, E^s)$ is called the fibre component of σ. We define the set $\boldsymbol{Lip}^u(\lambda)$ to be the set of all unstable sections whose fibre component s, has a fibre Lipschitz constant less than λ. We define $\boldsymbol{Lip}^s(\lambda)$ similarly.

7.1 The Unstable Manifold Theorem

Theorem 7.1 (The Unstable Manifold Theorem for normed vector bundles) *Let E^s and E^u be a pair of normed vector bundles over a common base space, H for which the splitting $E^s \oplus_H E^u$ is h_0-non-degenerate for $h_0 \geq 1$. Consider a hyperbolic linear fibre contraction T of the fibre rhombus $E^s \oplus_H E^u$ with hyperbolicity constant $0 < \lambda < 1$. If $\kappa > 1$, and $0 < \lambda\kappa < \bar{\lambda} < 1 \leq h_0 < \bar{h}_0$ then there exist positive constants ε, and δ, depending only on λ, κ, and $\bar{\lambda}$, for which the following is true.*

Let $R : H \to (0, \infty)$ be a κ-slowly varying function. Let f be a $C^{r+\beta}_\kappa$ metric bundle map of the κ-slowly varying disc bundle $\triangle_R E^s \oplus_H \triangle_R E^u$ to $E^s \oplus_H E^u$ as bundles over H. Denote the zero section of the bundle π by $\mathbf{0}_H$ and assume that the two sections, $f(\mathbf{0}_H)$ and $f^{-1}(\mathbf{0}_H)$, of $E^s \oplus_H E^u$ are also sections of $\triangle_R E^s \oplus_H \triangle_R E^u$. Furthermore, assume that

$$\rho = \max\left\{\frac{|f(\mathbf{0}_H(p))|_{h(p)}}{R(p)}, \quad \frac{|f^{-1}(\mathbf{0}_H(p))|_{h^{-1}(p)}}{R(p)}\right\} \leq \delta.$$

Choose $\bar{\rho}$ such that $\frac{\rho}{\delta} \leq \bar{\rho} \leq 1$, and let $r(p) = \bar{\rho}R(p)$.

If

$$Lip(f - T) \leq \varepsilon, \quad Lip(Df - T) \leq \varepsilon,$$
$$Lip\left(f^{-1} - T^{-1}\right) \leq \varepsilon, \quad Lip\left(Df^{-1} - T^{-1}\right) \leq \varepsilon$$

then the following three f invariant sections exist

1. $g^u : \triangle_R \boldsymbol{E}^u \to \triangle_R \boldsymbol{E}^s \oplus_H \triangle_R \boldsymbol{E}^u$,

2. $g^s : \triangle_R \boldsymbol{E}^s \to \triangle_R \boldsymbol{E}^s \oplus_H \triangle_R \boldsymbol{E}^u$,

3. $x : H \to \triangle_r \boldsymbol{E}^s \oplus_H \triangle_r \boldsymbol{E}^u$,

and satisfy the following conditions

1. $g^s \in \boldsymbol{Lip}^s\left(\bar{\lambda}\right)$, $g^u \in \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$,

2. g^s *and* g^u *are* $C^{r+\beta}_{nK(r+\beta)}$,

3. f *contracts* g^s, *and expands* g^u, *that is,* $f \circ g^u(\triangle_r \boldsymbol{E}^u) \supset g^u(\triangle_r \boldsymbol{E}^u)$ *and* $g^s(\triangle_r \boldsymbol{E}^s) \supset f \circ g^s(\triangle_r \boldsymbol{E}^s)$,

4. *the infinite intersection of the forward (inverse) iterates of* $\triangle_r \boldsymbol{E}^s \oplus_H \triangle_r \boldsymbol{E}^u$ *by* f *is the graph of* g^u (g^s), *that is*

$$\begin{aligned} g^u(\triangle_r \boldsymbol{E}^u) &= \bigcap_{n=0}^{\infty} f^n(\triangle_r \boldsymbol{E}^s \oplus_H \triangle_r \boldsymbol{E}^u), \quad \text{and} \\ g^s(\triangle_r \boldsymbol{E}^s) &= \bigcap_{n=0}^{\infty} f^{-n}(\triangle_r \boldsymbol{E}^s \oplus_H \triangle_r \boldsymbol{E}^u), \end{aligned}$$

5. *the sections* g^s *and* g^u *intersect at* x *which is the only* f *invariant section of* $\triangle_R \boldsymbol{E}^s \oplus_H \triangle_R \boldsymbol{E}^u$ *over* H,

6. g^s *and* g^u *are the stable and unstable manifolds of* x *in* $\triangle_R \boldsymbol{E}^s \oplus_H \triangle_R \boldsymbol{E}^u$, *that is, if* $p \in H$, $e_u \in (\triangle_r \boldsymbol{E}^u)_p$ *and* $e_s \in (\triangle_r \boldsymbol{E}^s)_p$ *then*

$$\partial\left(f^{-n} \circ g^u(e_u), f^{-n} \circ x(p)\right) \to 0, \quad \text{and} \quad \partial\left(f^n \circ g^s(e_s), f^n \circ x(p)\right) \to 0$$

as $n \to \infty$,

7. *there exists an* f*-invariant splitting,* $\boldsymbol{E}^s \oplus_H \boldsymbol{E}^u = \tilde{\boldsymbol{E}}^s \oplus_H \tilde{\boldsymbol{E}}^u$, *of the tangent space of the section* x *which is* $\bar{h}_0$*-non-degenerate. Moreover,* f *is hyperbolic at* x *with expansion and contraction constant* $\bar{\lambda}$ *with respect to this splitting.*

The definition of ρ given above might have made more sense as

$$\rho_0 = \max\left\{ \frac{|f(0_H(p))|_{h(p)}}{R(h(p))}, \quad \frac{|f^{-1}(0_H(p))|_{h^{-1}(p)}}{R(h^{-1}(p))} \right\}.$$

Since R is a κ-slowly varying function we know that $\frac{1}{\kappa} \leq \frac{R}{r} \leq \kappa$, and so the difference is slight. The form we have used will be more convenient in the body of the proof.

Proof: We will essentially follow Shub's proof of the unstable manifold theorem for a fixed point. Indeed this proof can be seen as essentially *an* implementation of the proof sketched by Pugh and Shub in [PS89]. Note however, that in their proof they assumed that there was *no* perturbation of the zero section of π. That is they assumed that $\rho \equiv 0$. It is *very* important for the proof of the *Weakly Shadowing* Stable manifold theorem, proven in Part III, that we allow the metric bundle map, f, to perturb this zero section, and hence *this proof* allows for the possibility that $\rho > 0$.

Since f is Lipschitz close to T and since T is a fibre contraction of $\tilde{\pi}_u$, we could hope to create a fibre contraction out of f itself. In fact there are many ways to do this, one way for each unstable section σ in $\boldsymbol{Lip}^u\left(\lambda\right)$.

Define $f_s = \tilde{\pi}_s \circ f$ and $f_u = \tilde{\pi}_u \circ f$. Given an unstable section $\sigma \in \boldsymbol{Lip}^u(\lambda)$ we can construct an *unstable base expansion* via the following diagram

$$\begin{tikzcd}
\triangle_R \boldsymbol{E}^u \arrow[r, "\sigma"] \arrow[d, dash] & \triangle_R \boldsymbol{E}^u \oplus_H \triangle_R \boldsymbol{E}^s \arrow[r, "f"] \arrow[d, dash] & \boldsymbol{E}^u \oplus_H \boldsymbol{E}^s \arrow[r, "\tilde{\pi}_u"] \arrow[d, dash] & \boldsymbol{E}^u \arrow[d, dash] \\
H \arrow[r, "Id"] & H \arrow[r, "h"] & H \arrow[r, "Id"] & H
\end{tikzcd}$$

That is we define

$$b_\sigma = b_{u,\sigma} = f_u \circ \sigma = \tilde{\pi}_u \circ f \circ \sigma.$$

We can then construct the following *unstable fibre contraction* out of f and this base expansion, $b_{u,\sigma}$, by defining

$$\begin{aligned}
F_\sigma\left(e_u + e_s\right) = F_{u,\sigma}\left(e_u + e_s\right) &= b_{u,\sigma}\left(e_u\right) + f_s\left(e_u + e_s\right), \\
&= \tilde{\pi}_u \circ f \circ \sigma\left(e_u\right) + \tilde{\pi}_s \circ f\left(e_u + e_s\right).
\end{aligned}$$

Finally, given that $F_{u,\sigma}$ is a fibre contraction, we can, as in the last two chapters, define the *unstable σ-Graph Transform* associated to $F_{u,\sigma}$ as

$$\Gamma_F(\tilde{\sigma}) = \Gamma_{u,F}(\tilde{\sigma}) = \Gamma_{u,F_\sigma}(\tilde{\sigma}) = F_{u,\sigma} \circ \tilde{\sigma} \circ b_{u,\sigma}^{-1}$$

for any $\tilde{\sigma} \in \boldsymbol{Lip}^u(\lambda)$.

While this is the proper form of a "Graph Transform", it is not quite enough to show the existence of the (unique) f-invariant unstable section g^u. This is because the section σ_{F_σ} need not be invariant under its *own* σ_{F_σ}-Graph Transform.

The following *unstable Graph Transform* is the correct Graph Transform to associate to f. It is defined as

$$\Gamma_f(\sigma) = \Gamma_{u,f}(\sigma) = \Gamma_{u,F_\sigma}(\sigma) = F_{u,\sigma} \circ \sigma \circ b_{u,\sigma}^{-1}$$

for any $\sigma \in \boldsymbol{Lip}^u(\lambda)$. The unstable section, g^u, which is the fixed point of Γ_f determines the *most* natural fibre contraction to associate to f, since *it is* invariant under its *own* g^u-Graph Transform. The Lipschitz part of this proof shows that there is only one natural way to make a fibre contraction out of f since there is only one unstable section which is invariant under its own Graph Transform. More importantly this unstable section is f-invariant. We will use the unstable Graph Transform to do this.

Since the conditions in this lemma are symmetric in time reversal, we obtain g^s via the stable Graph Transform in a similar fashion. Define $f_s^{-1} = \tilde{\pi}_s \circ f^{-1}$, and $f_u^{-1} = \tilde{\pi}_u \circ f^{-1}$. Given a stable section $\sigma \in \boldsymbol{Lip}^s(\lambda)$ we can construct a *stable base expansion* by defining

$$b_\sigma = b_{s,\sigma} = f_s^{-1} \circ \sigma = \tilde{\pi}_s \circ f^{-1} \circ \sigma.$$

We can then construct the following *stable fibre contraction* out of f and this base expansion, $b_{s,\sigma}$, by defining

$$F_\sigma(e_s + e_u) = F_{s,\sigma}(e_s + e_u) = b_{s,\sigma}(e_s) + f_u^{-1}(e_s + e_u).$$

Again, given that $F_{s,\sigma}$ is a fibre contraction, we can, as in the last section, define the *stable σ-Graph Transform* associated to $F_{s,\sigma}$ as

$$\Gamma_F(\tilde{\sigma}) = \Gamma_{s,F}(\tilde{\sigma}) = \Gamma_{s,F_\sigma}(\tilde{\sigma}) = F_{s,\sigma} \circ \tilde{\sigma} \circ b_{s,\sigma}^{-1}$$

for any $\bar{\sigma} \in \boldsymbol{Lip}^s(\lambda)$. The *stable Graph Transform* is

$$\Gamma_f(\sigma) = \Gamma_{s,f}(\sigma) = \Gamma_{s,F_\sigma}(\sigma)$$

for any $\sigma \in \boldsymbol{Lip}^s(\lambda)$.

We will assume the following defining conditions on ε, and δ and the auxiliary variable, $\bar{\varepsilon} > 0$.

$$(\lambda + 2\varepsilon)\kappa < \bar{\lambda} < 1 \tag{7.1a}$$
$$0 < \delta < \frac{1}{\kappa} - \lambda - \varepsilon \tag{7.1b}$$
$$0 < \delta < \frac{1}{\lambda} - \kappa - 2\varepsilon \tag{7.1c}$$
$$\bar{\varepsilon} = \frac{2\varepsilon h_0}{1 - \lambda - 2\varepsilon} < 1 \tag{7.1d}$$
$$h_0\left[\frac{1+\bar{\varepsilon}}{1-\bar{\varepsilon}}\right]^2 < \bar{h}_0 \tag{7.1e}$$

It is *very* important, for a complete understanding of both the statment and proof of this theorem, to note that the function, $r : H \to (0, \infty)$, is defined by $r(p) = \bar{\rho}R(p)$ for an *arbitrarily* fixed constant $\bar{\rho}$ for which $\frac{\varepsilon}{2} \le \bar{\rho} \le 1$.

7.2 Step 1: Existence due to Lipschitz estimates

7.2.1 Existence of g^u

Our aim in this section of the proof is to show that for each $\sigma \in \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$ the metric bundle map $F_{u,\sigma}$ is a fibre contraction. We begin by showing that the metric bundle map $b_{u,\sigma}$ is a fibre expansion.

Lemma 7.2 *For all σ in $\boldsymbol{Lip}^u\left(\bar{\lambda}\right)$, the metric bundle map $b_{u,\sigma}$ is a fibre expansion of $\triangle_r E^u$ into E^u. Moreover the fibre constant $\alpha_{u,\sigma}$ of $b_{u,\sigma}$ satisfies*

$$\alpha_{u,\sigma} \ge \frac{1}{\lambda} - \varepsilon.$$

Proof: In order to show that $b_\sigma = b_{u,\sigma}$ is a fibre expansion we need to show that it is a Lipschitz perturbation of $T_u = T|_{E^u} = \bar{\pi}_u \circ T \circ 0_{E^u}$. Recall that $b_\sigma = f_u \circ \sigma = \bar{\pi}_u \circ f \circ \sigma$, that T commutes with $\bar{\pi}_u$, and that $\bar{\pi}_u \circ \sigma = Id_{E^u_0}$. Hence we can write $b_\sigma - T_u = \bar{\pi}_u \circ (f - T) \circ \sigma$. This then implies that

$$Lip(b_\sigma - T_u) \le Lip(\bar{\pi}_u)\, Lip(f - T)\, Lip(\sigma) = Lip(f - T) \le \varepsilon,$$

where $\bar{\pi}_u$, $(f - T)$, and σ are interpreted as metric bundle maps as in the following diagram

$$\begin{tikzcd}
\triangle_R E^u \arrow[r, "\sigma"] \arrow[d, "\pi_u"'] & \triangle_R E^u \oplus_H \triangle_R E^s \arrow[r, "f - T"] \arrow[d, "\pi"'] & E^s \oplus_H E^u \arrow[r, "\pi_u"] \arrow[d, "\pi"'] & E^u \arrow[d, "\pi_u"'] \\
H \arrow[r, "Id"] & H \arrow[r, "h"] & H \arrow[r, "Id"] & H
\end{tikzcd}$$

In particular, since $\sigma \in Lip^u\left(\bar{\lambda}\right)$ and $\bar{\lambda} < 1$, we know that $Lip(\sigma) = 1$ *as a metric bundle map from* E^u *to* π.

Since $\varepsilon < \frac{1}{\lambda} < Lip_f\left(T_u^{-1}\right)^{-1}$, the Lipschitz inverse function theorem for fibre bundles 5.2 allows us to conclude that b_σ is a metric bundle homeomorphism of $\triangle_r E^u$ into E^u, and moreover that b_σ^{-1} is a Lipschitz metric bundle map with *fibre* Lipschitz constant

$$Lip_f\left(b_\sigma^{-1}\right) \le \frac{1}{\frac{1}{\lambda} - \varepsilon}.$$

This implies that the base constant $\alpha_{u,\sigma}$ of b_σ is

$$\alpha_{u,\sigma} \ge \frac{1}{\lambda} - \varepsilon.$$

To finish showing that b_σ is a fibre expansion we must show that $\triangle_r E^u \subset b_\sigma(\triangle_r E^u)$. Since $Lip_f(b_\sigma^{-1}) \le \frac{1}{1/\lambda - \varepsilon}$, we know that, for every p in H, $b_\sigma\left(B_{r(p),p}(E^u)\right)$ contains the ball of radius $r(p)/\lambda - \varepsilon$ about $b_\sigma(0) = f_u \circ \sigma(0)$, and hence it contains the ball of radius $\xi(p) = r(p)(1/\lambda - \varepsilon) - |b_\sigma(0)|_p$. Let s denote the fibre component of σ, then $b_\sigma(0) = f_u(0 + s(0))$. Since T is a bundle map of the bundle $\tilde{\pi}_u$, we have

$$\begin{aligned}
&|f_u(0+s(0))|_{h(p)} \\
&\quad \leq |f_u(0)|_{h(p)} + |f_u(0+s(0)) - f_u(0)|_{h(p)} \\
&\quad \leq |f_u(0)|_{h(p)} + |(f_u - T_u)(0+s(0)) - (f_u - T_u)(0)|_{h(p)} \\
&\qquad + |T_u(0+s(0))|_{h(p)} + |T_u(0)|_p \\
&\quad \leq |f(0)|_{h(p)} + \varepsilon r(p) + 0 + 0.
\end{aligned}$$

This implies that

$$\xi(p) = \frac{r(p)}{\lambda} - 2\varepsilon r(p) - |f(0)|_{h(p)}.$$

Condition 7.1c and the inequality $\rho \leq \bar{\rho}\delta$ then imply that $r(h(p)) < \xi(p)$. Since p in H was arbitrary we have shown that b_σ is a fibre expansion of $\triangle_r E^u$. ∎

We now want to show that $F_{u,\sigma}$ is itself a fibre contraction.

Lemma 7.3 *For all σ in $\boldsymbol{Lip}^u(\bar{\lambda})$, the metric bundle map $F_{u,\sigma}$ is a fibre contraction over the fibre expansion $b_{u,\sigma}$. Moreover the fibre constant $k_{u,\sigma}$ of $F_{u,\sigma}$ satisfies*

$$k_{u,\sigma} \leq \lambda + \varepsilon$$

Proof: Recall that we have defined the fibre contraction $F_\sigma = F_{u,\sigma}$ to be,

$$F_\sigma(e_u + e_s) = b_\sigma(e_u) + f_s(e_u + e_s).$$

To begin showing that F_σ is a fibre contraction over b_σ we must show that

$$F_\sigma(\triangle_r E^u \oplus_H \triangle_r E^s) \bigcap (\triangle_r E^u \oplus_H E^s) \subset \triangle_r E^u \oplus_H \triangle_r E^s.$$

Since b_σ is a fibre expansion it is enough to show that for the fibre component of F_σ and for every $p \in H$, $f_s\left(B_{r(p),p}(E^u \oplus_H E^s)\right) \subset B_{r(h(p)),h(p)}(E^s)$ where the ball, $B_{r(p),p}(E^u \oplus_H E^s)$, is taken relative to the box fibre norm of the bundle π. To do this we note that

$$\begin{aligned}
&|f_s(e_u + e_s)|_{h(p)} \\
&\quad \leq |f_s(e_u + e_s) - T_s(e_u + e_s)|_{h(p)} + |T_s(e_u + e_s)|_{h(p)} \\
&\quad \leq |(f_s - T_s)(e_u + e_s)|_{h(p)} + \lambda r(p)
\end{aligned}$$

$$
\begin{aligned}
&\leq \; |(f_s - T_s)(e_u + e_s) - (f_s - T_s)(0+0)|_{h(p)} \\
&\quad + |(f_s - T_s)(0+0)|_{h(p)} + \lambda r(p) \\
&\leq \; |f(0)|_{h(p)} + (\lambda + \varepsilon) r(p) .
\end{aligned}
$$

The final $|f(0)|_{h(p)}$ is, since $Lip(\bar{\pi}_u) = 1$ taken relative to the box fibre norm of the bundle π. Recall that $|f(0)|_{h(p)} \leq \rho R(p) \leq \delta r(p)$. Condition 7.1b then implies that $|f_s(e_u + e_s)|_{h(p)} < r(h(p))$.

Finally we want to estimate the fibre constant of F_σ. That is we want to estimate the fibre Lipschitz constant of F_σ in the fibres of the bundle $\bar{\pi}_u$. To do this consider the following inequality

$$
\begin{aligned}
&|F_\sigma(e_u + e_s) - F_\sigma(e_u + \bar{e}_s)|_{b_\sigma(e_u)} \\
&\quad = \; |f_s(e_u + e_s) - f_s(e_u + \bar{e}_s)|_{h(p)} \\
&\quad \leq \; |(f_s - T_s)(e_u + e_s) - (f_s - T_s)(e_u + \bar{e}_s)|_{h(p)} \\
&\qquad + |T_s(e_u + e_s) - T_s(e_u + \bar{e}_s)|_{h(p)} \\
&\quad \leq \; (\lambda + \varepsilon)|(e_u + e_s) - (e_u + \bar{e}_s)|_p \\
&\quad \leq \; (\lambda + \varepsilon)|e_s - \bar{e}_s|_p
\end{aligned}
$$

■

We have now shown that for any σ in $\boldsymbol{Lip}^u(\bar{\lambda})$ the fibre bundle map $F_{u,\sigma}$ is a fibre bundle contraction. We can now conclude that there is a unique section σ_{F_σ} which is invariant under the action of the "σ-Graph Transform". Unfortunately this unique section is not usually the one we want because it is not usually invariant under its "own" "Graph Transform". The next lemma shows that there is a unique section in $\boldsymbol{Lip}^u(\bar{\lambda})$ which is invariant under its own Graph Transform. We will call this section g^u.

Lemma 7.4 *The unstable Graph Transform $\Gamma_{u,f}$ maps $\boldsymbol{Lip}^u(\bar{\lambda})$ into itself, and, moreover it contracts distances in $\boldsymbol{Lip}^u(\bar{\lambda})$ by a factor of at least $(\lambda + 2\varepsilon)$. This implies that $\Gamma_{u,f}$ has a unique fixed point g^u.*

Proof: We start by showing that $\Gamma_f = \Gamma_{u,f}$ maps $\boldsymbol{Lip}^u(\bar{\lambda})$ into itself. It is important to note that $\boldsymbol{Lip}^u(\bar{\lambda})$ is a subset of the space of all sections of the bundle

$\tilde{\pi}_u : \triangle_r E^s \oplus_H \triangle_r E^u \to \triangle_r E^u$. Consider $\sigma \in Lip^u\left(\bar{\lambda}\right)$. This means that the fibre Lipschitz constant of the fibre component, s, of σ, is less than $\bar{\lambda}$. From the body of the proofs of lemmas 7.2 and 7.3, we know that $\Gamma_f(\sigma)(\triangle_r E^u) \subset \triangle_r E^s \oplus_H \triangle_r E^u$. Hence to show that Γ_f maps $Lip^u\left(\bar{\lambda}\right)$ into itself we only need to show that the fibre Lipschitz constant of the fibre component of $\Gamma_f(\sigma)$ is also less than $\bar{\lambda}$.

Recall that

$$\Gamma_f(\sigma) = F_\sigma \circ \sigma \circ b_\sigma^{-1} = Id + f_s \circ \sigma \circ b_\sigma^{-1},$$

and hence that the fibre component of $\Gamma_f(\sigma)$ is $f_s \circ \sigma \circ b_\sigma^{-1}$. Since $Lip_f(s) \leq \bar{\lambda}$, we know that $Lip(\sigma) \leq 1$. By arguments similar to those used in lemma 7.2, this implies that $Lip(f_s \circ \sigma - T_s \circ \sigma) \leq Lip(f - T) \leq \varepsilon$, and hence $Lip_f(f_s \circ \sigma) \leq \lambda + \varepsilon$. Lemma 7.2 and conditions 7.1a and 7.1c, then imply that

$$\begin{aligned} Lip_f(\tilde{\pi}_s \circ \Gamma_f(\sigma)) &\leq Lip_f(f_s \circ \sigma)\, Lip_f\left(b_\sigma^{-1}\right) \\ &\leq \frac{\lambda + \varepsilon}{\frac{1}{\mu} - \varepsilon} \\ &< \bar{\lambda}. \end{aligned}$$

We now want to show that Γ_f contracts distances in $Lip^u\left(\bar{\lambda}\right)$. Recall that the distance metric defined on the space of sections of $\tilde{\pi}_u : \triangle_R E^s \oplus_H \triangle_R E^u \to \triangle_r E^u$ is the sup norm of the E^s fibre norm of the fibre component of the section. We show that Γ_f contracts this distance metric in two steps.

In the first step we show that, by working in the fibre $\tilde{\pi}_u^{-1}(e_u)$, the $\tilde{\pi}_u$ *fibre* distance between an arbitrary point $e_u + e_s$ and the point $\sigma(e_u) = e_u + s(e_u)$ decreases by a factor of at least $(\lambda + 2\varepsilon)$ under the action of $F_\sigma = F_{u,\sigma}$. That is, for all σ in $Lip^u\left(\bar{\lambda}\right)$, p in H, e_s in $(\triangle_r E^s)_p$, and e_u in $(\triangle_r E^u)_p$ for which $f_u(e_u + e_s)$ lies in $(\triangle_r E^u)_{h(p)}$, the following inequality holds

$$|f_s(e_u + e_s) - \tilde{\pi}_s \circ \Gamma_f(\sigma) \circ f_u(e_u + e_s)|_{h(p)} \leq (\lambda + 2\varepsilon)\, |e_s - s(e_u)|_p .$$

To obtain this inequality, recall that $Lip_f(f_s) \leq (\lambda + \varepsilon)$, $Lip(f_u - T_u) \leq \varepsilon$, and that T commutes with $\tilde{\pi}_u$. Recall, also, that the fibre component of $\Gamma_f(\sigma)$ is $\tilde{\pi}_s \circ \Gamma_f(\sigma) = f_s \circ \sigma \circ b_\sigma^{-1}$. This allows us to note that

$$\begin{aligned} f_s(e_u + s(e_u)) &= f_s \circ \sigma \circ b_\sigma^{-1} \circ b_\sigma(e_u) \\ &= \tilde{\pi}_s \circ \Gamma_f(\sigma) \circ f_u(e_u + s(e_u)) \end{aligned}$$

More importantly, we have shown, above that the fibre Lipschitz constant of the fibre component of $\Gamma_f(\sigma)$ is less than one. Finally, let $p = \pi(e_u + e_s) = \pi_u(e_u) = \pi_s(e_s)$. With these considerations, we then have

$$\begin{aligned}
&|f_s(e_u + e_s) - \tilde{\pi}_s \circ \Gamma_f(\sigma) \circ f_u(e_u + e_s)|_{h(p)} \\
&\quad\leq |f_s(e_u + e_s) - f_s(e_u + s(e_u))|_{h(p)} \\
&\qquad + |f_s(e_u + s(e_u)) - \tilde{\pi}_s \circ \Gamma_f(\sigma) \circ f_u(e_u + e_s)|_{h(p)} \\
&\quad\leq (\lambda + \varepsilon)|e_s - s(e_u)|_p \\
&\qquad + |\tilde{\pi}_s \circ \Gamma_f(\sigma)(f_u(e_u + s(e_u))) - \tilde{\pi}_s \circ \Gamma_f(\sigma)(f_u(e_u + e_s))|_{h(p)} \\
&\quad\leq (\lambda + \varepsilon)|e_s - s(e_u)|_p + |f_u(e_u + e_s) - f_u(e_u + s(e_u))|_{h(p)} \\
&\quad\leq (\lambda + \varepsilon)|e_s - s(e_u)|_p \\
&\qquad + |(f_u - T_u)(e_u + e_s) - (f_u - T_u)(e_u + s(e_u))|_{h(p)} \\
&\qquad + |T_u(e_u + e_s) - T_u(e_u + s(e_u))|_{h(p)} \\
&\quad\leq (\lambda + \varepsilon)|e_s - s(e_u)|_p + \varepsilon|e_s - s(e_u)|_p + |\tilde{\pi}_u \circ T(e_s - s(e_u))|_{h(p)} \\
&\quad\leq (\lambda + 2\varepsilon)|e_s - s(e_u)|_p + 0.
\end{aligned}$$

Now to finish showing that $\Gamma_f(\sigma)$ contracts distances in $\boldsymbol{Lip}^u(\bar{\lambda})$ by a factor of at least $(\lambda + 2\varepsilon)$, let σ and $\tilde{\sigma}$ be two sections in $\boldsymbol{Lip}^u(\bar{\lambda})$. Let s and $\tilde{s}$ denote their respective fibre components, and let z be a point in $(\triangle_r \boldsymbol{E}^u)_p$. By applying the previous inequality to $\tilde{\sigma}$ at the point $e_u + e_s = (\sigma \circ b_\sigma^{-1})(z)$, we have the following inequality

$$|\tilde{\pi}_s \circ \Gamma_f(\sigma)(z) - \tilde{\pi}_s \circ \Gamma_f(\tilde{\sigma})(z)|_{h(p)} \leq (\lambda + 2\varepsilon)\left|s\left(b_\sigma^{-1}(z)\right) - \tilde{s}\left(b_\sigma^{-1}(z)\right)\right|_p.$$

By taking the supremum over all z in $\triangle_r \boldsymbol{E}^u$ we get the required result.

We have now shown that Γ_f is a contraction mapping of $\boldsymbol{Lip}^u(\bar{\lambda})$ into itself. Since $\boldsymbol{Lip}^u(\bar{\lambda})$ is a closed subspace of the space of sections of $\tilde{\pi}_u : \triangle_r \boldsymbol{E}^u \oplus_H \triangle_r \boldsymbol{E}^s \to \triangle_r \boldsymbol{E}^u$, and since this space of sections is complete, the contraction mapping principle allows us to conclude that Γ_f has a unique fixed point which we denote g^u. ∎

Since the zero section, $0_{\triangle_r \boldsymbol{E}^u}$, of $\tilde{\pi}_u : \triangle_r \boldsymbol{E}^u \oplus_H \triangle_r \boldsymbol{E}^s \to \triangle_r \boldsymbol{E}^u$ is contained in $\boldsymbol{Lip}^u(\bar{\lambda})$, the previous lemma also allows us to conclude that $\Gamma^n_{u,f}(0_{\triangle_r \boldsymbol{E}^u})$ is a

Cauchy sequence in $\boldsymbol{Lip}^u\left(\bar{\lambda}\right)$ and that

$$g^u = \lim_{n\to\infty} \Gamma^n_{u,f}\left(0_{\triangle_r E^u}\right).$$

Corollary 7.5 *$|g^u(e_u)| \leq \bar{\rho}R(e_u)$ for all $e_u \in \triangle_r E^u$.*

Proof: Let s^u denote the fibre component of g^u. Then it is enough to show that, $|s^u(e_u)|_p \leq r(p)$, for all $p \in H$ and all $e_u \in (\triangle_r E^u)_p$.

Let $g_n = \Gamma^n_{u,f}(0_{\triangle_R E^u})$ and let s_n denote the fibre component of g_n. Since $\lim_{n\to\infty} g_n = g^u$, it is in fact enough to show that, $|s_n(e_u)|_p \leq r(p)$ for all $p \in H$, $e_u \in (\triangle_r E^u)_p$, and all $n > 0$.

We prove this latter statement by induction. It is certainly true for $g_0 = 0_{\triangle_R E^u}$. Now consider $n > 0$ and assume that the statement is true for $n-1$.

Using the definition of the unstable Graph Transform, we know that

$$g_n = \Gamma_{u,f}(g_{n-1}) = Id_{E^u} + f_s \circ g_{n-1} \circ b^{-1}_{g_{n-1}},$$

and so, $s_n = f_s \circ g_{n-1} \circ b^{-1}_{g_{n-1}}$. Since $g_0 \in \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$, we have $g_m \in \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$ for all $m > 0$. In particular, $g_{n-1} \in \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$. Hence we know from lemma 7.2 that $\triangle_r E^u \subset b_{u,g_{n-1}}(\triangle_r E^u)$. The body of lemma 7.3 showed that g^u is a section of $\bar{\pi}_u : \triangle_r E^s \oplus_H \triangle_r E^u \to \triangle_r E^u$ and hence $g^u(\triangle_r E^u) \subset \triangle_r E^s \oplus_H \triangle_r E^u$. We can conclude the proof, by noticing that the body of the proof of lemma 7.3 also showed that $f_s(\triangle_r E^s \oplus_H \triangle_r E^u) \subset \triangle_r E^s$. ∎

7.2.2 Existence of g^s

The previous three lemmas have all dealt with showing the existence of the g^u through the use of the unstable Graph Transform $\Gamma_{u,f}$. Since the conditions of the main lemma are symmetric with respect to time reversal, the arguments showing the existence of the g^s through the use of the $\Gamma_{s,f}$ are similar. For convenience in later use we will summarize the dual results in the next few lemmas.

Lemma 7.6 *For all σ in $\boldsymbol{Lip}^s\left(\bar{\lambda}\right)$ the metric bundle map $b_{s,\sigma}$ of the bundle π_s, is a fibre expansion of $\triangle_r E^s$ into E^s. Moreover the base constant $\alpha_{s,\sigma}$ of $b_{s,\sigma}$ satisfies*

$$\alpha_{s,\sigma} \geq \frac{1}{\lambda} - \varepsilon.$$

Lemma 7.7 *For all σ in $\boldsymbol{Lip}^s\left(\bar{\lambda}\right)$, the metric bundle map $F_{s,\sigma}$ is a fibre contraction over the fibre expansion $b_{s,\sigma}$. Moreover the fibre constant $k_{s,\sigma}$ of $F_{s,\sigma}$ satisfies*

$$k_{s,\sigma} \leq \lambda + \varepsilon.$$

Lemma 7.8 *The stable Graph Transform $\Gamma_{s,f}$ maps $\boldsymbol{Lip}^s\left(\bar{\lambda}\right)$ into itself, and, moreover $\Gamma_{s,f}$ contracts distance in $\boldsymbol{Lip}^s\left(\bar{\lambda}\right)$ by a factor of at least $(\lambda + 2\varepsilon)$. This implies that $\Gamma_{s,f}$ has a unique fixed point g^s.*

Again since the zero section, $0_{\Delta_r \boldsymbol{E}^s}$, of the bundle $\bar{\pi}_s : \Delta_r \boldsymbol{E}^s \oplus_H \Delta_r \boldsymbol{E}^u \longrightarrow \Delta_r \boldsymbol{E}^s$ is contained in $\boldsymbol{Lip}^s\left(\bar{\lambda}\right)$, the previous lemma also allows us to conclude that $\Gamma^n_{s,f}\left(0_{\Delta_r \boldsymbol{E}^s}\right)$ is a Cauchy sequence in $\boldsymbol{Lip}^s\left(\bar{\lambda}\right)$ and that

$$g^s = \lim_{n \to \infty} \Gamma^n_{s,f}\left(0_{\Delta_r \boldsymbol{E}^s}\right).$$

Corollary 7.9 *$|g^s(e_s)| \leq \bar{\rho} R(e_s)$ for all $e_s \in \Delta_r \boldsymbol{E}^s$.*

7.2.3 g^s and g^u are stable and unstable manifolds

We are now interested in showing that the sections, g^u and g^s, are the unstable and stable manifolds.

Lemma 7.10 *f expands g^u and contracts g^s. That is $f \circ g^u(\Delta_r \boldsymbol{E}^u) \supset g^u(\Delta_r \boldsymbol{E}^u)$ and $g^s(\Delta_r \boldsymbol{E}^s) \subset f^{-1} \circ g^s(\Delta_r \boldsymbol{E}^s)$.*

Proof: Since b_{u,g^u} is a fibre expansion, $z_0 \in \Delta_r \boldsymbol{E}^u$ implies that $z_1 = b^{-1}_{u,g^u}(z_0) \in \Delta_r \boldsymbol{E}^u$. By using the fact that g^u is a fixed point of $\Gamma_{u,f}$ we have

$$\begin{aligned} g^u(z_0) &= f \circ g^u \circ b^{-1}_{u,g^u}(z_0) \\ &= f \circ g^u(z_1) \end{aligned}$$

The statement involving g^s follows similarly. ■

Lemma 7.11 *The infinite intersection of the forward (inverse) iterates of $\Delta_r \boldsymbol{E}^s \oplus_H \Delta_r \boldsymbol{E}^u$ by the metric bundle map, f, is the graph of g^u (g^s). That is $g^u(\Delta_r \boldsymbol{E}^u) = \bigcap_{n=0}^{\infty} f^n(\Delta_r \boldsymbol{E}^s \oplus_H \Delta_r \boldsymbol{E}^u)$ and $g^s(\Delta_r \boldsymbol{E}^s) = \bigcap_{n=0}^{\infty} f^{-n}(\Delta_r \boldsymbol{E}^s \oplus_H \Delta_r \boldsymbol{E}^u)$*

Proof: Fix n and consider a point $\bar{e}_u + \bar{e}_s \in \triangle_r E^s \oplus_H \triangle_r E^u$ for which $f^n(\bar{e}_u + \bar{e}_s) = e_u + e_s \in \triangle_r E^s \oplus_H \triangle_r E^u$. Let $p = \pi(\bar{e}_u + \bar{e}_s)$, $p = \pi(e_u + e_s)$, and finally let s^u denote the fibre component of the section g^u. Since g^u is the fixed point of $\Gamma_{u,f}$, the body of the proof of lemma 7.4 implies that

$$\begin{aligned} |e_s - s^u(e_u)|_p &\leq (\lambda + 2\varepsilon)^n |\bar{e}_s - s^u(\bar{e}_u)|_{\bar{p}} \\ &\leq (\lambda + 2\varepsilon)^n r(\bar{p}) \end{aligned}$$

Since $\bar{p} = h^n(p)$, and r is a κ-slowly varying sequence we know

$$|e_s - s^u(e_u)|_p \leq [(\lambda + 2\varepsilon)\kappa]^n r(p).$$

Condition 7.1a then implies that if $e_u + e_s \in \triangle_r E^s \oplus_H \triangle_r E^u$ is a point for which $f^{-n}(e_u + e_s) \in \triangle_r E^s \oplus_H \triangle_r E^u$ for all positive n then $e_s = s^u(e_u)$. Hence $g^u(\triangle_r E^u)$ contains the above infinite intersection. Since $g^u(\triangle_r E^u) \subset \triangle_r E^s \oplus_H \triangle_r E^u$ and $g^u(\triangle_r E^u) \subset f \circ g^u(\triangle_r E^u)$ the other direction follows. The dual statement is similar. ■

Lemma 7.12 *If e and $\bar{e}$ are contained in the same fibre of $\triangle_r E^u$ then $d(f^{-n} \circ g^u(e), f^{-n} \circ g^u(\bar{e})) \to 0$ as $n \to \infty$. Dually, if e and $\bar{e}$ are contained in the same fibre of $\triangle_r E^s$ then $d(f^n \circ g^s(e), f^n \circ g^s(\bar{e})) \to 0$ as $n \to \infty$.*

Proof: We will, again, only prove the unstable case since the stable case is similar. Consider $p \in H$ and $e, \bar{e} \in (\triangle_r E^u)_p$. Since g^u is the fixed point of $\Gamma_{u,f}$, we know that $f^{-n} \circ g^u = g^u \circ b^{-n}_{g^u}$. Working in the projection onto E^u we have

$$\begin{aligned} \left|f_u^{-n} \circ g^u(e) - f_u^{-n} \circ g^u(\bar{e})\right|_{h^{-n}(p)} &\leq \left|b_{g^u}^{-n}(e) - b_{g^u}^{-n}(\bar{e})\right|_{h^{-n}(p)} \\ &\leq \left(\frac{1}{1/\lambda - \varepsilon}\right)^n |e - \bar{e}|_p. \end{aligned}$$

Similarly working in the projection onto E^s we have

$$\begin{aligned} \left|f_s^{-n} \circ g^u(e) - f_s^{-n} \circ g^u(\bar{e})\right|_{h^{-n}(p)} &\leq \left|s^u \circ b_{g^u}^{-n}(e) - s^u \circ b_{g^u}^{-n}(\bar{e})\right|_{h^{-n}(p)} \\ &\leq Lip_f(s^u)\left(\frac{1}{1/\lambda - \varepsilon}\right)^n |e - \bar{e}|_p. \end{aligned}$$

Since $g^u \in \boldsymbol{Lip}^u(\bar{\lambda})$, $Lip_f(s^u) \leq \bar{\lambda} < 1$. Condition 7.1c implies that $\frac{1}{1/\lambda - \varepsilon} < 1$, hence the right hand side of both inequalities converge to zero as n goes to ∞.

Since f is a π bundle map of $E^s \oplus_H E^u$, the appropriate distance metric to use in the fibres of $E^s \oplus_H E^u$ is the box norm of the fibre norms of E^u and E^s respectively. The two inequalities proven above, then imply that $d(f^{-n} \circ g^u(e), f^{-n} \circ g^u(\bar{e})) \mapsto 0$ as $n \mapsto \infty$. ∎

7.2.4 Existence of x

We are now interested in showing that the pair of sections, g^u and g^s, intersect at a unique section, x, of $\pi : \triangle_r E^s \oplus_H \triangle_r E^u \to H$, and moreover that this section is f-invariant. In order to do this, we define, for the pair of sections, $\sigma^u \in Lip^u(\bar{\lambda})$ and $\sigma^s \in Lip^s(\bar{\lambda})$, $\sigma^u \# \sigma^s$ to be the section of sets of intersection points of σ^u and σ^s in the fibres of $\triangle_r E^s \oplus_H \triangle_r E^u$.

Lemma 7.13 *The section of sets, $\sigma^u \# \sigma^s$, is a section of single points. Moreover, this section, $x = g^u \# g^s$, is an f-invariant section of the fibre bundle $\pi : \triangle_r E^s \oplus_H \triangle_r E^u \to H$.*

Proof: Let s^u and s^s denote the fibre components of g^u and g^s respectively. Consider $p \in H$. The sections g^u and g^s are defined over the whole of $(\triangle_R E^u)_p$ and $(\triangle_R E^s)_p$ respectively. For $\bar{\rho} < 1$, Corollaries 7.5 and 7.9 imply that over the *proper* subsets $(\triangle_r E^u)_p \subset (\triangle_R E^u)_p$, and $(\triangle_r E^s)_p \subset (\triangle_R E^s)_p$, the graphs of g^u and g^s must lie in the box $(\triangle_r E^s \oplus_H \triangle_r E^u)_p$. This implies that there must be at least one transversal intersection of g^u and g^s inside the closure of $(\triangle_r E^s \oplus_H \triangle_r E^u)_p$.

Now chose any point, x, in this intersection, and place a pair of transversal cones based at x with slopes of $\bar{\lambda}$ in the stable and unstable directions respectively. Since $\bar{\lambda} < 1$, these cones do not intersect *except* at the point x. Since $g^u \in Lip^u(\bar{\lambda})$ and $g^s \in Lip^s(\bar{\lambda})$ and since, moreover they intersect at x, the sections g^s and g^u must lie in the stable and unstable cones based at x. This implies that the point *x is the only* point of intersection of g^u and g^s.

Since $p \in H$ was arbitrary, there exists exactly one point of intersection of g^u and g^s in each fibre of $\triangle_R E^s \oplus_H \triangle_R E^u$ which is moreover contained in $\triangle_r E^s \oplus_H \triangle_r E^u$. Together these intersection points form a section of the bundle $\triangle_R E^s \oplus_H \triangle_R E^u$. Let x denote this section.

In the body of lemma 7.4 we showed that $Lip_f(f_s \circ \sigma) \leq \lambda + \bar{\lambda}$ for all $\sigma \in \boldsymbol{Lip}^u(\bar{\lambda})$. This means that the fibre Lipschitz constant of the fibre component of $f \circ g^u$, is at most $\bar{\lambda}$. Lemma 7.10 shows that $f \circ g^s(\triangle_R \boldsymbol{E}^s) \subset g^s(\triangle_R \boldsymbol{E}^s)$ and hence the fibre Lipschitz constant of the fibre component of $f \circ g^s$, is also at most $\bar{\lambda}$. Hence by placing stable and unstable cones based at $f(g^u \# g^s) = f \circ g^u \# f \circ g^s$ with slopes $\bar{\lambda}$, we can again show that the sections, $f \circ g^s$ and $f \circ g^u$ intersect at exactly one point, namely $f(g^u \# g^s)$.

Again By lemma 7.10 we know that $f(g^u \# g^s)$ is contained in $g^s(\triangle_R \boldsymbol{E}^s)$. Since $f(g^u(\triangle_R \boldsymbol{E}^u)) \supset g^u(\triangle_R \boldsymbol{E}^u)$, we see that the unique intersection $g^u \# g^s$ of the sections g^u and g^s is equal to the unique intersection $f(g^u \# g^s)$ of the sections $f \circ g^u$ and $f \circ g^s$. That is $f(g^u \# g^s) = g^u \# g^s$. This means that the section, $x = g^u \# g^s$ is f-invariant. ∎

While the following lemma is not strictly required for the main theorems in this thesis, it does provide a useful characterization of the section x as a Cauchy sequence of the zero section of $\boldsymbol{E}^s \oplus_H \boldsymbol{E}^u$. To simplify the notation, let $g_n^u = \Gamma_{u,f}^n(0_{\triangle_R \boldsymbol{E}^u})$ and $g_n^s = \Gamma_{s,f}^n(0_{\triangle_R \boldsymbol{E}^s})$, and finally, let s_n^u and s_n^s denote the fibre components of g_n^u and g_n^s respectively.

Lemma 7.14 *Let* $0 < m_u, m_s$, *then* $g_{m_u}^u \# g_{m_s}^s$ *is a section of the bundle* $\pi : \triangle_r \boldsymbol{E}^s \oplus_H \triangle_r \boldsymbol{E}^u \to H$. *Moreover, for all* $0 < m_u, m_s$, $g_{m_u}^u \# g_{m_s}^s$, *is a Cauchy sequence with respect to* m_u, *with respect to* m_s, *and jointly with respect to* (m_u, m_s).

Proof: We begin by showing that for fixed m_u, $g_{m_u}^u \# g_{m_s}^s$, is a Cauchy sequence with respect to m_s. Consider $m_s, \bar{m}_s > 0$. Fix $p \in H$. To simplify the following discussion we will work in the fibre $(\triangle_r \boldsymbol{E}^s \oplus_H \triangle_r \boldsymbol{E}^u)_p$. As in figure 7.1, define

$$
\begin{aligned}
u(p) &= g_{m_u}^u \# g_{m_s}^s(p) = u_u + u_s = s_{m_s}^s(u_s) + u_s, \\
v(p) &= v_u + v_s = s_{\bar{m}_s}^s(u_s) + u_s, \\
w(p) &= g_{m_u}^u \# g_{\bar{m}_s}^s = w_u + w_s.
\end{aligned}
$$

Since $g_{m_u}^u \in \boldsymbol{Lip}^u(\bar{\lambda})$, we know that $|w_s - u_s|_p \leq \bar{\lambda}\,|w_u - u_u|_p$. Similarly, since $g_{\bar{m}_s}^s \in \boldsymbol{Lip}^s(\bar{\lambda})$, we know that $|w_u - v_u|_p \leq \bar{\lambda}\,|w_s - v_s|_p$. Let $z_u = v_u - u_u$. Since

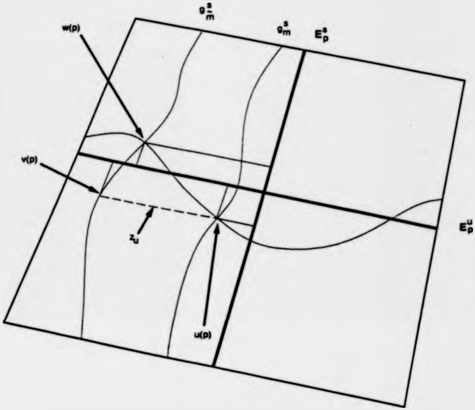

Figure 7.1: Diagram used to prove that for fixed m_u, $g^u_{m_u} \# g^s_{m_s}$, is a Cauchy sequence with respect to m_s.

g_m^s is a Cauchy sequence we can make $|z_u|_p$ arbitrarily small by choosing m_s, $\tilde{m}_s$ large enough. Combining the previous two inequalities we have

$$|w_u - u_u|_p - |z_u|_p \leq |w_u - u_u - z_u|_p \leq \bar{\lambda}\,|w_s - u_s|_p \leq \bar{\lambda}^2\,|w_u - u_u|_p.$$

Hence, since $\bar{\lambda} < 1$, we have

$$\begin{aligned} |w_u - u_u|_p &\leq \frac{|z_u|_p}{\left(1-\bar{\lambda}^2\right)} \\ |w_s - u_s|_p &\leq \frac{\bar{\lambda}\,|z_u|_p}{\left(1-\bar{\lambda}^2\right)}. \end{aligned}$$

Hence in the box norm in the fibre $(\triangle_R E^s \oplus_H \triangle_R E^u)_p$ the distance between $g^u_{m_u} \# g^s_{m_s}$ and $g^u_{m_u} \# g^s_{\tilde{m}_s}$ can be made arbitrarily small by choosing $|z_u|_p$ small, and this can be done by choosing m_s and $\tilde{m}_s$ large enough.

In fact since g_m^s is a Cauchy sequence in the space of sections of the bundle $\bar{\pi}_s$, the norm of z_u can be made arbitrarily small *independently* of the specific, fixed, $g^u_{m_u}$. In particular this means that for all $\varepsilon > 0$, there exists an $M > 0$ such that for all $m_u > 0$, and all $m_s, \tilde{m}_s > M$

$$\left|g^u_{m_u} \# g^s_{m_s} - g^u_{m_u} \# g^s_{\tilde{m}_s}\right|_p \leq \varepsilon$$

Similar arguments can be used to show that $g^u_{m_u} \# g^s_{m_s}$ is a Cauchy sequence with respect to m_u for all $m_s > 0$.

Now, to show that $g^u_{m_u} \# g^s_{m_s}$ is a Cauchy sequence for both m_u, and m_s jointly, consider $\varepsilon > 0$. Choose $M_s > 0$ for which for all $m_u > 0$, and all $m_s, \tilde{m}_s > M_s$

$$\left|g^u_{m_u} \# g^s_{m_s} - g^u_{m_u} \# g^s_{\tilde{m}_s}\right|_p \leq \frac{\varepsilon}{2}.$$

Similarly choose $M_u > 0$ for which for all $m_u, \tilde{m}_u > M_u$, and all $m_s > 0$,

$$\left|g^u_{m_u} \# g^s_{m_s} - g^u_{\tilde{m}_u} \# g^s_{m_s}\right|_p \leq \frac{\varepsilon}{2}.$$

Let M be the maximum of M_u and M_s and consider $m_u, \tilde{m}_u, m_s, \tilde{m}_s > M$. Then we have

$$\begin{aligned} &\left|g^u_{m_u} \# g^s_{m_s} - g^u_{\tilde{m}_u} \# g^s_{\tilde{m}_s}\right|_p \\ &\quad\leq \left|g^u_{m_u} \# g^s_{m_s} - g^u_{\tilde{m}_u} \# g^s_{m_s}\right|_p + \left|g^u_{\tilde{m}_u} \# g^s_{m_s} - g^u_{\tilde{m}_u} \# g^s_{\tilde{m}_s}\right|_p \\ &\quad\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

■

7.3 Step 2: C^1 continuity of g^u and g^s

Having shown the existence of $g^u \in \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$ and $g^s \in \boldsymbol{Lip}^s\left(\bar{\lambda}\right)$ we would like to show that g^u and g^s are as differentiable as f. Since F_{u,g^u} is a fibre contraction over the fibre expansion b_{u,g^u} we could hope that Pugh and Shub's C^r_*-Section theorem might apply. Unfortunately, since b_{u,g^u} is only as continuous as g^u we can not show that F_{u,g^u} is even a 1-fibre contraction. We must deduce the higher differentiablity of g^u from a different fibre contraction.

Define $\mathbf{L}^u\left(\bar{\lambda}\right)$ to be the set of linear sections of the normed vector bundle $\bar{\pi}_u$ for which the fibre Lipschitz constant of the fibre component is less than or equal to $\bar{\lambda}$. Via restriction, any linear section of the normed vector bundle $\bar{\pi}_u$ can be viewed as a linear section of any varying disc bundle formed from the bundle $\bar{\pi}_u$. Clearly $\mathbf{L}^u\left(\bar{\lambda}\right) \subset \boldsymbol{Lip}^u\left(\bar{\lambda}\right)$ where $\mathbf{L}^u\left(\bar{\lambda}\right)$ and $\boldsymbol{Lip}^u\left(\bar{\lambda}\right)$ are sets of sections of any arbitrary disc bundle formed from the bundle $\bar{\pi}_u$.

Let $U_\varepsilon = \{S \in \mathrm{L}\left(\boldsymbol{E}^s \oplus_H \boldsymbol{E}^u, \boldsymbol{E}^s \oplus_H \boldsymbol{E}^u\right) \mid \|S - T\| < \varepsilon\}$. Consider $S \in U_\varepsilon$. Since S is a linear map of the bundle π, if $\sigma \in \mathbf{L}^u\left(\bar{\lambda}\right)$ is a linear section of the bundle $\bar{\pi}_u$, then $S \circ \sigma$ is also a linear section of the bundle $\bar{\pi}_u$ and hence $\Gamma_{u,S}$ is well defined and maps $\mathbf{L}^u\left(\bar{\lambda}\right)$ into itself. Lemma 7.4 then implies that the Lipschitz constant of $\Gamma_{u,S}$ is less than or equal to $(\lambda + 2\varepsilon)\,\kappa$. Moreover, since composition and inversion are continuous on the space of linear maps, $\Gamma : U_\varepsilon \times \mathbf{L}^u\left(\bar{\lambda}\right) \to \mathbf{L}^u\left(\bar{\lambda}\right)$ is a continuous map.

Recall that $\triangle_{\bar{\lambda}}\mathrm{L}\left(\boldsymbol{E}^u, \boldsymbol{E}^s\right)$ is the normed varying disc bundle of linear mappings from the fibres of $\boldsymbol{E}^u$ to the fibres of $\boldsymbol{E}^s$ for which the supremum of the operator norm $\|L\|$ is less than or equal to $\bar{\lambda}$. To show that g^u is C^1, we construct a fibre contraction map of the Whitney sum bundle $\boldsymbol{E}_u \oplus \mathrm{L}\left(\boldsymbol{E}_u, \boldsymbol{E}_s\right)$ to itself. The hope is that the invariant section of this fibre contraction will be the derivative of g^u. Since $\mathbf{L}^u\left(\bar{\lambda}\right)$ is isometrically isomorphic to $\triangle_{\bar{\lambda}}\mathrm{L}\left(\boldsymbol{E}^u, \boldsymbol{E}^s\right)$ we can use the Graph Transform Γ to define the required continuous fibre contraction F as follows

$$
\begin{aligned}
F^u &: \triangle_R \boldsymbol{E}^u \oplus \triangle_{\bar{\lambda}}\mathrm{L}\left(\boldsymbol{E}_u, \boldsymbol{E}_s\right) \to \boldsymbol{E}_u \oplus \mathrm{L}\left(\boldsymbol{E}_u, \boldsymbol{E}_s\right) \\
F^u\left(e_u + L\right) &= b_{u,g^u} + \Gamma_{D_{g^u(e_u)}f}\left(L\right).
\end{aligned}
$$

Fix $p \in H$ and $e_u \in \left(\triangle_R \boldsymbol{E}^u\right)_p$. If we write $D_{g^u(e_u)}f$ with respect to the splittings $E^s_p \oplus E^u_p$ and $E^s_{h(p)} \oplus E^u_{h(p)}$ as follows

$$D_{g^u(e_u)} f = \begin{matrix} A_{e_u} & B_{e_u} \\ C_x e_u & D_{e_u} \end{matrix}$$

where

$$\begin{aligned} A_{e_u} &\in \mathbf{L}\left(E^s_p, E^s_{h(p)}\right) \\ B_{e_u} &\in \mathbf{L}\left(E^u_p, E^s_{h(p)}\right) \\ C_{e_u} &\in \mathbf{L}\left(E^s_p, E^u_{h(p)}\right) \\ D_{e_u} &\in \mathbf{L}\left(E^u_p, E^u_{h(p)}\right) \end{aligned}$$

Moreover, if $\left\|D_{g^u(e_u)} f - T\right\| < \varepsilon$ and $\left\|D_{g^u(e_u)} f^{-1} - T^{-1}\right\| < \varepsilon$ then we know that

$$\begin{aligned} \|A_{e_u}\|, \left\|D^{-1}_{e_u}\right\| &\leq \lambda + \varepsilon \text{ and} \\ \|B_{e_u}\|, \|C_{e_u}\| &\leq \varepsilon \end{aligned}$$

With this notation we can write the fibre contraction, F^u, as

$$F^u(e_u + L) = b_{u,g^u} + [B_{e_u} + A_{e_u} L]\,[D_{e_u} + C_{e_u}]^{-1}.$$

Written this way, it is easy to see that

$$|F^u(0_{\triangle_R E^u})| = \sup_{e_u \in \triangle_R E^u} |F^u(e_u + 0)| = \sup_{e_u \in \triangle_R E^u} \left|B_{e_u} D^{-1}_{e_u}\right|.$$

If $\|D_e f - T\|$, $\|D_e f^{-1} - T^{-1}\| < \varepsilon$ for all $e \in \triangle_R E^s \oplus_H \triangle_R E^u$, then $|F^u(0_{\triangle_R E^u})| < \varepsilon$.

This discussion proves

Lemma 7.15 *If $\|D_e f - T\| < \varepsilon$ for all $e \in \triangle_R E^s \oplus_H \triangle_R E^u$ then F^u is a continuous fibre contraction of $\triangle_R E^u \oplus \triangle_1 \mathbf{L}(E_u, E_s)$ over the fibre expansion b_{u,g^u} of $\triangle_R E^u$ with a fibre constant of $(\lambda + 2\varepsilon)$. Hence Γ_{u,F^u} has a unique continuous F^u-invariant section τ_{u,F^u}, and moreover, $|\tau_{u,F^u}| \leq \frac{\varepsilon}{1-\lambda-2\varepsilon}$.*

Having proven that F^u is well defined and has an F^u-invariant section τ^u, we must now show, in the following lemma, that τ^u is the section of tangent planes to the section g^u.

Lemma 7.16 *The section τ^u is the section of tangent planes to the section g^u.*

Proof: Recall that in step 3 of the C^r_λ-Section Theorem (Theorem 6.2) we had to prove essentially the same thing. While this proof is essentially the same, we have a slight additional difficulty to worry about. In step 3 of theorem 6.2, the Graph Transform, and in particular the fibre expansion, b, did not depend on the section being transformed. In the present case this is not true. In the present Graph Transforms, Γ_F *and* Γ_{Df}, the base expansion, $b_{u,\sigma}$, *depends* on the section, σ, which is being transformed.

As in step 3 of theorem 6.2, let s and t denote the fibre components of g^u and τ^u respectively. Consider $\bar{\xi} \in \triangle_r E^u$ and let $p = \pi_1\left(\bar{\xi}\right)$. Define

$$\begin{aligned} l_{\bar{\xi}} &: (\triangle_r E^u)_p \to (\triangle_r E^s)_p \\ l_{\bar{\xi}}(e) &= s\left(\bar{\xi}\right) + t\left(\bar{\xi}\right)(e), \end{aligned}$$

recall that $T_{\bar{\xi}}(\triangle_r E^u)_p \simeq (\triangle_r E^u)_p$, and let $\lambda_{\bar{\xi}}$ denote the local section of $(\triangle_r E^u \oplus_H \triangle_r E^s)\big|_{(\triangle_r E^u)_p}$ defined by $\lambda_{\bar{\xi}}\left(\bar{\xi}+h\right) = \bar{\xi}+h+l_{\bar{\xi}}(h)$ for all $h \in (\triangle_r E^u)_p$. By construction, $l_{\bar{\xi}}$ is the fibre component of $\lambda_{\bar{\xi}}$ in $\triangle_r E^u \oplus \triangle_r \mathbf{L}(E^u, E^s)$.

The section τ^u of the normed vector bundle $\bar{\pi}_u : E_u \oplus \mathbf{L}(E_u, E_s) \to E^u$ is the section of tangent planes to the section g^u iff for all $p \in H$ and $\bar{\xi} \in (E^u)_p$ the two local sections g^u and $\lambda_{\bar{\xi}}$ are tangent at $\bar{\xi}$ iff for all $p \in H$ and $\bar{\xi} \in (E^u)_p$ the slope of the local section $g^u - \lambda_{\bar{\xi}}$ at $\bar{\xi}$ is zero.

Consider two local sections σ and $\bar{\sigma}$ in $Lip^u\left(\bar{\lambda}\right)$ which agree at the point $\bar{\xi}_0 \in \triangle_r E^1$. Since the local sections are contained in $Lip^u\left(\bar{\lambda}\right)$, they both have slopes less than or equal to $\bar{\lambda}$. Lemma 7.4 then implies that $\Gamma_f(\sigma)$ and $\Gamma_f(\bar{\sigma})$ are local sections of $\triangle_r E^u \oplus_H \triangle_r E^s$ contained in $Lip^u\left(\bar{\lambda}\right)$. Since $\sigma(\xi_0) = \bar{\sigma}(\xi_0)$, we know that $b_{u,\sigma}(\xi_0) = b_{u,\bar{\sigma}}(\xi_0)$. Let $\xi_1 = b_{u,\sigma}(\xi_0)$. This implies in turn, that $\Gamma_f(\sigma)$ and $\Gamma_f(\bar{\sigma})$ agree at the point $\bar{\xi}_1$. Let s and $\bar{s}$ denote the fibre components of σ and $\bar{\sigma}$ respectively. Let $\Delta_{\bar{\xi}_0} = \Delta_{\bar{\xi}_0}(\sigma, \bar{\sigma}) = Lip_{\bar{\xi}_0}(s - \bar{s})$, and finally let $\Delta_{\bar{\xi}_1} = \Delta_{\bar{\xi}_1}(\Gamma_f(\sigma), \Gamma_f(\bar{\sigma}))$. Let $p_0 = \bar{\pi}_u\left(\bar{\xi}_0\right)$, and $p_1 = \bar{\pi}_u\left(\bar{\xi}_1\right)$, then lemmas 7.2 and 7.4 and condition 7.1c imply that

$$\begin{aligned} \Delta_{\bar{\xi}_1} &= \lim_{\epsilon \to 0} \sup_{\xi_1 \in B_\epsilon(\bar{\xi}_1) \cap \pi_1^{-1}(p_1)} \frac{\left| f\sigma b_\sigma^{-1}(\xi_1) - f\bar{\sigma} b_{\bar{\sigma}}^{-1}(\xi_1)\right|_{p_1}}{\left|\xi_1 - \bar{\xi}_1\right|_{p_1}} \\ &\leq \lim_{\epsilon \to 0} \sup_{\xi_0 \in B_\epsilon(\bar{\xi}_0) \cap \pi_1^{-1}(p_0)} (\lambda + 2\varepsilon) \frac{\left|s(\xi_0) - \bar{s}(\xi_0)\right|_{p_0}}{\left|\xi_0 - \bar{\xi}_0\right|_{p_0}} \frac{\left|\xi_0 - \bar{\xi}_0\right|_{p_0}}{\left|\xi_1 - \bar{\xi}_1\right|_{p_1}} \end{aligned}$$

$$\leq \ (\lambda + 2\varepsilon)\, \Delta_{\tilde{\xi}_0},$$

where $\xi_1 = b_\sigma(\xi_0)$.

We are now interested in showing that the unstable Graph Transform, Γ_f, act naturally on the set of $\lambda_{\tilde{\xi}}$. In the following arguments, it is important to note that since the local section, $\lambda_{\tilde{\xi}}$, is defined only in a single fibre of $\bar{\pi}_u : \Delta_r \boldsymbol{E}^u \oplus_H \Delta_r \boldsymbol{E}^s \to \boldsymbol{E}^u$, we are really *only* working with maps of Banach spaces. Consider $\tilde{\xi}_1 \in \Delta_r \boldsymbol{E}^u$ and let $\tilde{\xi}_0 = b_{g^u}^{-1}\left(\tilde{\xi}_1\right)$. We want to show that

$$Lip_{\tilde{\xi}_1}\left(\Gamma_f\left(\lambda_{\tilde{\xi}_0}\right)\right) = Lip_{\tilde{\xi}_1}\left(\lambda_{\tilde{\xi}_1}\right).$$

From the definition of $\lambda_{\tilde{\xi}_0}$ we know that $g^u\left(\tilde{\xi}_0\right) = \lambda_{\tilde{\xi}_0}\left(\tilde{\xi}_0\right)$. This implies that $b_{g^u}\left(\tilde{\xi}_0\right) = b_{\lambda_{\tilde{\xi}_0}}\left(\tilde{\xi}_0\right)$ hence we know that $b_{\lambda_{\tilde{\xi}_0}}^{-1}\left(\tilde{\xi}_1\right) = \tilde{\xi}_0$. It is then an easy calculation to show that $\Gamma_f\left(\lambda_{\tilde{\xi}_0}\right)\left(\tilde{\xi}_1\right) = \Gamma_f\left(g^u\right)\left(\tilde{\xi}_1\right)$.

While we do not yet know that g^u is C^1, we do know that, by construction, $\lambda_{\tilde{\xi}_0}$ is C^1. So we can consider $D_{\tilde{\xi}_1} b_{\lambda_{\tilde{\xi}_0}}^{-1}$. Since τ^u is a section of $\Delta_r \boldsymbol{E}^u \oplus \Delta_{\bar{\lambda}} \mathrm{L}\left(\boldsymbol{E}_u, \boldsymbol{E}_s\right)$ we know that $\left\|l_{\tilde{\xi}_0}\right\| \leq \bar{\lambda} < 1$ and so $D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0}$ is invertible. Since $Lip(Df - T) \leq \varepsilon$, lemma 7.2 implies that $b_{u, D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0}}$ is invertible. Hence we can apply the inverse function theorem to show that

$$D_{\tilde{\xi}_1} b_{\lambda_{\tilde{\xi}_0}}^{-1} = \left[\bar{\pi}_u \circ D_{g^u(\tilde{\xi}_0)} f \circ D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0}\right]^{-1} = b_{u, D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0}}^{-1}.$$

We can now use Taylor's theorem about the point $\tilde{\xi}_1$ to approximate

$$\begin{aligned}
&\Gamma_f\left(\lambda_{\tilde{\xi}_0}\right)\left(\tilde{\xi}_1 + h\right) \\
&\quad = \ \Gamma_f\left(\lambda_{\tilde{\xi}_0}\right)\left(\tilde{\xi}_1\right) + D_{\tilde{\xi}_1}\Gamma_f\left(\lambda_{\tilde{\xi}_0}\right)(h) + O\left(|h|_{p_0}^2\right) \\
&\quad = \ \Gamma_f\left(g^u\right)\left(\tilde{\xi}_1\right) + D_{\tilde{\xi}_1} f \circ \lambda_{\tilde{\xi}_0} \circ b_{\lambda_{\tilde{\xi}_0}}^{-1}(h) + O\left(|h|_{p_0}^2\right) \\
&\quad = \ \Gamma_f\left(g^u\right)\left(\tilde{\xi}_1\right) + \left[D_{g^u(\tilde{\xi}_0)} f \circ D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0} \circ D_{\tilde{\xi}_1} b_{\lambda_{\tilde{\xi}_0}}^{-1}\right](h) + O\left(|h|_{p_0}^2\right) \\
&\quad = \ \Gamma_f\left(g^u\right)\left(\tilde{\xi}_1\right) + \left[D_{g^u(\tilde{\xi}_0)} f \circ D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0} \circ b_{u, D_{\tilde{\xi}_0}\lambda_{\tilde{\xi}_0}}^{-1}\right](h) + O\left(|h|_{p_0}^2\right) \\
&\quad = \ \Gamma_f\left(g^u\right)\left(\tilde{\xi}_1\right) + \Gamma_{Df}\left(\tau^u\right)\left(\tilde{\xi}_1\right)(h) + O\left(|h|_{p_0}^2\right) \\
&\quad = \ \lambda_{\tilde{\xi}_1}\left(\tilde{\xi}_1 + h\right) + O\left(|h|_{p_0}^2\right).
\end{aligned}$$

Now to conclude the proof, for $p \in H$, let

$$\Delta(p) = \sup\left\{\Delta_{\tilde{\xi}}\left(\lambda_{\tilde{\xi}}, g^u\right) \ \mid \ \tilde{\xi} \in \left(\Delta_r \boldsymbol{E}^u\right)_p\right\}.$$

The naturality of the action of Γ_f on $\lambda_{\bar{\xi}}$ implies that $\Delta_{\bar{\xi}_1}\left(\Gamma_f\left(\lambda_{\bar{\xi}_0}\right), \Gamma_f(g^u)\right) = \Delta_{\bar{\xi}_1}\left(\lambda_{\bar{\xi}_1}, g^u\right)$, and hence that $\Delta(h(p)) \le (\lambda + 2\varepsilon)\Delta(p)$. Since τ^u is uniformly bounded, the slope of $\lambda_{\bar{\xi}}$ is uniformly bounded. This and the fact that the slope of g^u is uniformly bounded implies that $\Delta(p)$ is a bounded function. If there exists a $p \in H$ for which $\Delta(p) \ne 0$, then $\Delta(h^{-n}(p)) \ge (\lambda + 2\varepsilon)^{-n}\Delta(p)$. Since this quantity grows too fast for $\Delta(p)$ to be bounded, we have that $\Delta(p) \equiv 0$, and hence τ^u is the section of tangent planes to g^u as required. ∎

Since F^u is a continuous bundle map, we know that τ^u is a continuous section. Since τ^u is the section of tangent planes to the section g^u, we see that g^u is C^1 *in the fibres* of E^u.

7.4 Hyperbolicity of x

We now consider the hyperbolicity of f at the section x. Since f is C^1_u, the required splitting comes from the tangent planes of g^u and g^s at x. The contraction and expansion essentially comes from the contraction of g^s and expansion of g^u by the action of f.

Lemma 7.17 *There exists an $\bar{h}_0$-non-degenerate splitting, $E^s \oplus_H E^u = \bar{E}^s \oplus_H \bar{E}^u$, with respect to which f is hyperbolic at x with a hyperbolicity constant of $\bar{\lambda}$.*

Proof: To show that f is hyperbolic at x we must produce, for each $p \in H$, a splitting of the tangent space of $x(p)$ which is preserved by $D_x f$.

Let l^u denote the fibre component of τ^u, and consider $p \in H$. For $\bar{\xi} \in (\Delta_r E^u)_p$, define $\lambda^u_{\bar{\xi}}(e_u) = e_u + l^u\left(\bar{\xi}\right)(e_u)$. The map, $\lambda^u_{\bar{\xi}}$ is then a linear section of the bundle $\hat{\pi}_u : (E^s \oplus_H E^u)_p \to (E^u)_p$. Recall that the last section showed that $D_{\bar{\xi}} g^u = \lambda^u_{\bar{\xi}}$. Note that the definitions of $\lambda^u_{\bar{\xi}}$ in this section and the definition of $\lambda_{\bar{\xi}}$ in the previous section, differ slightly. For $\bar{\xi} \in \Delta_r E^s$ define $\lambda^s_{\bar{\xi}}$ in an analogous fashion.

Let $x_u = \hat{\pi}_u \circ x$, and $x_s = \hat{\pi}_s \circ x$. Consider $p \in H$, and define $\bar{E}^u = \left\{\lambda^u_{x_u(p)}(e_u) \mid e_u \in (E^u)_p\right\}$. Similarly, define $\bar{E}^s = \left\{\lambda^s_{x_s(p)}(e_s) \mid e_s \in (E^s)_p\right\}$. Since $l^u \in \mathrm{L}^u\left(\bar{\lambda}\right)$, $l^s \in \mathrm{L}^s\left(\bar{\lambda}\right)$, and $\bar{\lambda} < 1$, we know that for all $p \in H$, the tangent space at $x(p)$ of the fibre $(E^s \oplus_H E^u)_p$ is, $T_{x(p)}(E^s \oplus_H E^u)_p \simeq (E^s \oplus_H E^u)_p \simeq (\bar{E}^u \oplus_H \bar{E}^s)_p$.

We want to show that

$$D_{x(h^{-1}(p))}f\left((\tilde{E}^{s})_{h^{-1}(p)}\right) = (\tilde{E}^{s})_{p}. \tag{7.2}$$

Consider $p \in H$ and $e_{s} \in (\tilde{E}^{s})_{p}$. Recall that $b_{l^{s}}(x(p)) = \pi\ uD_{x(p)}f \circ l^{s}(x_{s}(p)) : (E^{s})_{p} \to (E^{s})_{p}$. Let $v_{s} = l^{s}(x_{s}(p))(e_{s})$, and $\tilde{v}_{s} = \left(l^{s} \circ b_{l^{s}}^{-1}(x(h^{-1}(p)))\right)(e_{s})$. Then

$$\begin{aligned}
v_{s} &= l^{s}(x_{s}(p))(e_{s}) \\
&= \Gamma_{Df}(l^{s})(x_{s}(p))(e_{s}) \\
&= D_{x(h^{-1}(p))}f \circ l^{s} \circ b_{l^{s}}^{-1}\left(x\left(h^{-1}(p)\right)\right)(e) \\
&= D_{x(h^{-1}(p))}f\tilde{v}_{s}.
\end{aligned}$$

Since $b_{l^{s}}(x(p))$ is a linear isomorphism, equality 7.2 follows. A similar argument shows that $D_{x(h^{-1}(p))}f\left((\tilde{E}^{u})_{h^{-1}(p)}\right) = (\tilde{E}^{u})_{p}$.

Lemma 7.15 and its dual together imply that $\left|\lambda^{u}_{x_{u}}\right|, \left|\lambda^{s}_{x_{s}}\right| \le \frac{\varepsilon}{1-\lambda-2\varepsilon}$. Conditions 7.1d, and 7.1e along with lemma 4.2 then imply that the splitting, $E^{s} \oplus_{H} E^{u} = \tilde{E}^{s} \oplus_{H} \tilde{E}^{u}$ is $\bar{h}_{0}$-non-degenerate.

To see that f is hyperbolic at x we note that the above argument shows that $D_{x(h^{-1}(p))}f(\tilde{v}_{s}) = v_{s}$. Hence $\tilde{v}_{s} = D_{x(p)}f^{-1}(v_{s})$. Since $l^{s} \in \mathbf{L}^{s}\left(\bar{\lambda}\right)$, we know that $|\tilde{v}_{s}|_{h^{-1}(p)} = \left|b_{l^{s}}^{-1}(x(p))(e)\right|_{h^{-1}(p)}$ and $|v_{s}|_{p} = |e_{s}|_{p}$. Since $Lip\left(b_{l^{s}}^{-1}(x(p))\right) \le \frac{1}{1/\bar{\lambda}-\varepsilon} \le \bar{\lambda}$, we know that $\left|D_{x(p)}f^{-1}(v_{s})\right|_{h(p)} \le \bar{\lambda}\,|v_{s}|_{p}$. The dual statment is proven similarly. ∎

7.5 Step 4: C^{k}_{κ} continuity of g^{u} and g^{s}

The last step in proving Theorem 7.1 is to show that g^{u} and g^{s} are $C^{r+\beta}_{\kappa,\theta(r+\beta)}$. Recall that lemma 7.15 showed that F^{u} is a 0-fibre contraction with a fibre constant of $(\lambda + 2\varepsilon)$ and a base constant of $1/\bar{\lambda}$. Since $(\lambda + 2\varepsilon)\,\bar{\lambda}^{s}\kappa^{\theta(r-1+\beta)} < 1$ for all positive s, we see that F^{u} is an $(r-1+\beta)$-fibre contraction iff F^{u} is $C^{r-1+\beta}_{\kappa}$. Since F^{u} is constructed from the derivative of f and the section g^{u}, the fibre contraction F^{u} is $C^{r-1+\beta}_{\kappa}$ iff g^{u} is $C^{r-1+\beta}_{\kappa}$.

We have shown in step 2, that g^{u} is C^{1}. This means that F^{u} is a C^{1}_{κ} 1-fibre contraction. Hence r^{u} is $C^{1}_{\kappa\theta(1)}$ and so g^{u} is $C^{2}_{\kappa\theta(1)}$. This, in turn, means that F^{u}

is a $C^2_{\kappa\theta(1)}$ 2-fibre contraction. Hence r^u is $C^2_{\kappa\theta(2)}$ and so g^u is $C^3_{\kappa\theta(3)}$. And so on. Continuing this argument we see that g^u is $C^{r+3}_{\kappa\theta(r+3)}$. ■

Part III

Pseudo-hyperbolic Pseudo-orbits and Shadowing

The theory which we will develop in this part of the thesis is a weakly hyperbolic version of Anosov's Stability lemma [Ano70, Kat81, Shu87].

The central idea of Anosov's Stability lemma, as stated by Katok [Kat81] or Shub [Shu87], is that if we are given an action of a homeomorphism, h, on an topological space, X, and we are given an injection, i of X into the manifold M for which the set $i(X)$ is contained in a neighbourhood of a uniformly hyperbolic set, and moreover for which the injection makes the action of h on X and the action of f on $i(X)$ "almost" commute, *then* there is an injection, j which is close to i which makes the action of h on X *commute* with the action of f on $j(X)$.

This idea is best represented by the pair of diagrams

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "h"'] & M \arrow[d, "f"] \\ X \arrow[r, "i"] & M \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "h"'] & M \arrow[d, "f"] \\ X \arrow[r, "j"] & M \end{tikzcd}$$

which "almost" and actually commute respectively. The fact that the first diagram "almost" commutes means that the point $f \circ i(x)$ and the point $i \circ h(x)$ need not be equal *but* they must be close with respect to some uniform distance. This is the essentially the idea behind a *family* of pseudo-orbits. The topological space X acts as an index space which replaces the more common indexing by the integers used in the definition of a *single* pseudo-orbit. The action of h on the space X acts as the index function which replaces the more common "next integer" function.

The central idea in Anosov's Stability lemma is then that any family of pseudo-orbits which lie in a neighbourhood of a uniformly hyperbolic invariant set is shadowed by another invariant set whose dynamics is defined by the indexing function h. Note that the original uniformly hyperbolic invariant set forms the support for the pseudo-orbit to ensure that the pseudo-orbit can be shadowed. In order to distinguish between them we will generally call this pair of invariant sets the supporting and shadowing invariant sets respectively.

We need to extend this idea in two ways. Firstly, we would like to apply this lemma to weakly hyperbolic invariant sets. Secondly, we would like to obtain an estimate of the hyperbolicity of the resulting shadowing invariant set.

The lack of a hyperbolicity estimate in Anosov's Stability lemma is merely an oversight. In Anosov's original setting, that of Anosov diffeomorphisms, and in the standard Axiom-A setting, we know by other arguments, that the shadowing invariant set is as hyperbolic as the supporting invariant set. In the case of an Anosov diffeomorphism, *all* of the points in the manifold are uniformly hyperbolic with the same bounds. In the case of an Axiom-A diffeomorphism, *any* invariant set is uniformly hyperbolic with the same bounds.

We can extend Anosov's Stability lemma to encompass weakly hyperbolic invariants sets for two reasons. Firstly, it is *essentially* a *local* result. The pseudo-orbit is "close" to the supporting invariant set and the shadowing invariant set is "close" to the pseudo-orbit which is as a consequence "close" to the supporting invariant set. Secondly, by using a "metric" similar to the one Pesin used to prove his Stable manifold theorem for weakly hyperbolic invariant sets, we can similarly change the weak hyperbolicity into a uniform hyperbolicity.

Unfortunately, the "metric" which we use is really a countable family of metrics, or alternatively is a single metric on the disjoint union, $\boldsymbol{M}$, of a countable family of copies of the original manifold, M. This means that with respect to this "metric" our invariant sets and pseudo-orbits tend to "hop" between copies of the original manifold in $\boldsymbol{M}$. Fortunately, since Anosov's Stability lemma is essentially a local result, we can best view this "hopping" more properly as the action of a *fibre* bundle map on $T\boldsymbol{M}$ which is related to the original map f.

This suggests that the most natural way of making our extensions to Anosov's stability lemma is by explicitly formulating the *concept* of a (family of) pseudo-orbit(s). Since an invariant set is a special case of a pseudo-orbit it is then most natural to formulate the concept of a hyperbolic (or even pseudo-hyperbolic) *pseudo-orbit*. Note that our formulation of a hyperbolic pseudo-orbit encompasses precisely the fibre bundle structure with which it is most natural to state and prove our extension of Anosov's Stability lemma.

Having explicitly formulated the concept of a pseudo-hyperbolic pseudo-orbit, it is then most natural to break Anosov's rather monolithic stability lemma into a number of separate sublemmas each of which formulates an intuitively interesting aspect of the "theory of pseudo-orbits". In particular our theory states that

- given any weakly pseudo-hyperbolic pseudo-orbit of M, we can find a pseudo-hyperbolic pseudo-orbit of $\boldsymbol{M}$ *and* a metric of $\boldsymbol{M}$ with respect to which the new pseudo-hyperbolic pseudo-orbit is *uniformly* pseudo-hyperbolic,
- given any uniformly pseudo-hyperbolic pseudo-orbit of $\boldsymbol{M}$ there exists a unique splitting with respect to which the pseudo-orbit is *hyperbolic*,
- any uniformly hyperbolic invariant set has a neighbourhood, in $\boldsymbol{M}$, for which any other pseudo-orbit of $\boldsymbol{M}$ which is contained in this neighbourhood is a uniformly pseudo-hyperbolic pseudo-orbit,
- any uniformly hyperbolic pseudo-orbit is shadowed by a uniformly hyperbolic invariant set.

There are six chapters in this part of the thesis. The first two chapters define and explore the elementary properties of pseudo-orbits and pseudo-hyperbolic pseudo-orbits respectively. The final four chapters contain the proofs of the above four main parts of our theory of pseudo-hyperbolic pseudo-orbits. The statement and proof of *our* extension of Anosov's Stability lemma, which we choose to call the Weak Shadowing Stable Manifold theorem, can be found in the only chapter of Part IV.

Chapter 8

Pseudo-Orbits and the Classifying manifold, $\boldsymbol{M}$

The Weak Shadowing Stable Manifold theorem is stated in terms of (weakly) pseudo-hyperbolic pseudo-orbits. The next two chapters are devoted to the definition of exactly what these objects are. This chapter will deal with the definition and properties of pseudo-orbits. The next chapter will cover the definition of (pseudo) hyperbolicity of pseudo-orbit.

While it is not at first sight obvious, the Section theorems and the Stable Manifold theorem proven in Part II, are *very powerful* theorems. *The* power of these theorems is actually largely contained in the fact that they place very mild conditions on the structure of the base space H. In fact, they only require that the base space be paracompact. Essentially, the concept of a pseudo-hyperbolic pseudo-orbit is a bundle version of the concept of a hyperbolic invariant set and is precisely what is required to make use of these mild conditions.

Our aim in this chapter is to define factored pseudo-orbits of the classifying manifold, $\boldsymbol{M}$.

The (pseudo)-hyperbolic pseudo-orbits used in the rest of this thesis, are generally internally "classified" according to the "weakness" of their "immediate" hyperbolicity, i.e. by which "hyperbolic block" contains the given point of the given pseudo-orbit. Since there is, in general, one hyperbolic block associated to each non-negative integer, it is convenient to consider *a* manifold, $\boldsymbol{M}$, which is formed of a countably infinite number of disjoint copies of the original manifold M. We call the manifold $\boldsymbol{M}$ the classifying manifold.

While the actual object which is of interest in our theory is a pseudo-orbit of the *original* manifold, the techniques which we apply to prove our shadowing results only apply to pseudo-orbits of the classifying manifold. Fortunately, it is very easy to associate a pseudo-orbit of $\boldsymbol{M}$ to any (pseudo)-hyperbolic pseudo-orbit of M. However, since the real object of interest is the pseudo-orbit of M and *its* shadow, we need to ensure that associated to any pseudo-orbit of $\boldsymbol{M}$ there is a corresponding pseudo-orbit of M. This is precisely what a *factored* pseudo-orbit does. A factored pseudo-orbit of $\boldsymbol{M}$ consists of an (unfactored) pseudo-orbit of $\boldsymbol{M}$ which factors over a pseudo-orbit of M.

It is important that the (unfactored) pseudo-orbit of $\boldsymbol{M}$ *factors* over the pseudo-orbit of M. There can be, and in general is, more than one point of the pseudo-orbit of $\boldsymbol{M}$ which corresponds to any given point of the pseudo-orbit of M. Indeed the maximally shifted closure of a pseudo-orbit, to be defined below, is in a sense the largest possible cover of any given pseudo-orbit of M.

Since the concept of a factored pseudo-orbit of $\boldsymbol{M}$ consists of many layers, this chapter will consist of a sequence of simple definitions and related properties which together build to the final definition of a factored pseudo-orbit.

For the rest of this chapter we will assume that we are given a single, fixed, compact Riemannian manifold, M, together with a single, fixed, $C^{r+\gamma}$ diffeomorphism, f, which maps M to itself.

8.1 Uniform (unclassified) pseudo-orbits

Recall that a subset, Λ, of M is an *f-invariant set* if $f^{-1}(\Lambda) = \Lambda = f(\Lambda)$. The simplest invariant set is a single orbit, $\{f^n(x)\}_{-\infty}^{\infty}$ for some $x \in M$.

The usual definition of a pseudo-orbit is a generalization of a *single* orbit. It will actually be more convenient for our purposes to work with *families* of pseudo-orbits. A *α-pseudo orbit*, $\mathfrak{A}$, for the manifold M and the diffeomorphism f, is a triple $\mathfrak{A}(X, h, i)$, where X is a set, h is a bijection of X, and i is an injection of X into M for which the diagram

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "h"'] & M \arrow[d, "f"] \\
X \arrow[r, "i"] & M
\end{tikzcd}
$$

α-commutes, that is $d(i \circ h(x), f \circ i(x)) < \alpha$ for all $x \in X$. We call X, i, and h the index set, injection, and bijection respectively. A C^0 α-pseudo orbit, is an α-pseudo orbit for which X is a metric space and the functions h and i are C^0. For $1 \leq r$, a C^r pseudo-orbit is a C^0 pseudo-orbit $\mathfrak{A}(X, h, i)$ for which there exists a C^r *manifold* $\bar{X}$, C^r diffeomorphism, $\bar{h} : \bar{X} \to \bar{X}$, and a C^r map $\bar{i} : \bar{X} \to M$ for which $X \subset \bar{X}$, $h = \bar{h}\big|_X$, and $i = \bar{i}\big|_X$. A ***closed pseudo-orbit*** of M and f is a pseudo-orbit of M and f for which the set $i(X)$ is a closed subset of M.

This definition of a pseudo-orbit of M and f is inherently asymmetrical. Given a pseudo-orbit, $\mathfrak{A}(X, h, i)$, of M and f, define the ***inverse pseudo-orbit*** of M and f^{-1} associated to $\mathfrak{A}$ by $\mathfrak{A}^{-1}(X, h^{-1}, i)$. The inherent asymmetry of the above definition of a pseudo-orbit is most easily seen by the fact that the inverse pseudo-orbit, $\mathfrak{A}^{-1}$, of an α-pseudo orbit, $\mathfrak{A}$, need not be an α-pseudo orbit. However, if we let K denote the Lipschitz constant of f^{-1} then $\mathfrak{A}^{-1}$ *will be* a $K\alpha$-pseudo orbit whenever $\mathfrak{A}$ is an α-pseudo orbit.

In the next chapter we will define a constant, $\mathfrak{r}_M$, which is related to the injectivity radius of the original Riemannian metric of M. All pseudo-orbits in this thesis will be $\frac{\mathfrak{r}_M}{K}$-pseudo orbits with respect to the original metric where $K = \max\{1, Lip(f^{-1})\}$.

Given two pseudo-orbits. $\mathfrak{A}(X, h, i)$ and $\mathfrak{B}\left(\bar{X}, \bar{h}, \bar{i}\right)$, of M and f, $\mathfrak{B}$ is a ***sub-pseudo-orbit*** of $\mathfrak{A}$, if $\bar{X} \subset X$, $\bar{h} = h\big|_{\bar{X}}$, and $\bar{i} = i\big|_{\bar{X}}$. Note that since $\mathfrak{B}$ is itself a pseudo-orbit, the set $\bar{X}$ is h-invariant.

More generally, the pseudo-orbit $\mathfrak{B}$ is *embedded* in the pseudo-orbit $\mathfrak{A}$ if there exists an injective function $k_e : \bar{X} \to X$ which makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "h"] & X \arrow[dr, "i"] & \\ & & M \\ \tilde{X} \arrow[uu, "k_e"] \arrow[r, "\tilde{h}"] & \tilde{X} \arrow[uu, "k_e"] \arrow[ur, "\tilde{i}"'] & \end{tikzcd}$$

Dually, the pseudo-orbit $\mathfrak{B}$ is a *factor* of the pseudo-orbit $\mathfrak{A}$ if there exists an surjective function $k_f : X \to \tilde{X}$ which makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dd, "k_f"'] & X \arrow[dr, "i"] \arrow[dd, "k_f"'] & \\ & & M \\ \tilde{X} \arrow[r, "\tilde{h}"] & \tilde{X} \arrow[ur, "\tilde{i}"'] & \end{tikzcd}$$

The pseudo-orbits, $\mathfrak{A}$ and $\mathfrak{B}$. are *conjugate* if k_f is a homeomorphism. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^0 (C^r) pseudo-orbits then we assume that k_e and k_f are continuous (C^r).

For a given pseudo-orbit $\mathfrak{A}$, one of its most important properties is its *perturbation constant* which we will denote by $\mathfrak{p}(\mathfrak{A})$. We define the perturbation constant as $\mathfrak{p}(\mathfrak{A}) = \sup_{x \in X} d(i \circ h(x), f \circ i(x))$. A pseudo-orbit with a zero perturbation constant is an *invariant set*. The important relationship between a pseudo-orbit and an invariant set is that of shadowing. An invariant set $\mathfrak{B}\left(X, h, \tilde{i}\right)$ *β-shadows* a pseudo-orbit $\mathfrak{A}(X, h, i)$ if $d\left(i(x), \tilde{i}(x)\right) \leq \beta$ for all $x \in X$.

There are essentially two canonical examples of a (family of) pseudo-orbit(s). The first corresponds to the more usual definition of a "pseudo-orbit", namely the index space, X, is taken to be the integers together with the discrete topology, and $h(n) = n+1$. Since X has the discrete topology, any injection from X into M is continuous. The constraint that the homeomorphism h and the diffeomorphism f must α-commute via the injection i, is just the requirement that $i(X)$ must be an α-pseudo orbit for f in the usual sense.

The other canonical example of a pseudo-orbit, which is this time a *family* of pseudo-orbits, is that of a closed f-invariant set Λ relative to a diffeomorphism g which is C^0 close to f. In this case we take the index set, X, to be the set Λ together with the natural subspace topology of Λ as a subspace of M. The indexing bijection, h, is then the restriction of f to Λ, and the indexing injection, i, is just the canonical injection of Λ into M. If f and g are both C^r diffeomorphisms and M is compact, then the α-commuting condition is equivalent to the condition that $d(f(x), g(x)) \leq \alpha$ for all $x \in \Lambda$. In this case, since X can be considered as a subspace of the manifold M and both h and i are restrictions of C^r functions, we note that the pseudo-orbit is a C^r pseudo-orbit.

8.2 Classified pseudo-orbits

In the previous section, the definitions of pseudo-orbits, perturbation constants and shadowing were all defined *uniformly* relative to the original manifold M. We are now interested in defining non-uniform pseudo-orbits. To do this we require a few additional definitions.

A *classification* of X is a function $\chi : X \to \mathbb{Z}^+$. We can use this classification function of X to define a partition and a gradation of X as follows, for each $0 \leq n$,

- define the n^{th} *partition*, X_n, as $X_n = \chi^{-1}(n)$,
- define the n^{th} *gradation*, P_n, as $P_n = \bigcup_{i=0}^{n} X_i = \chi^{-1}(\{0, \ldots, n\})$.

Note that for $0 \leq n < m$, $X_n \bigcap X_m = \emptyset$, and $P_n \subset P_m$. From this definition we know that $X = \bigcup_{n=0}^{\infty} X_n = \bigcup_{n=0}^{\infty} P_n$. Given a sequence (R_n), and a classification of X, then we can define a function $R : X \to \mathbb{R}$ by $R(x) = R_{\chi(x)}$.

A pseudo-orbit, $\mathfrak{A}(X, h, i)$, together with a classification, χ, of X is a ***classified pseudo-orbit***, denoted $\mathfrak{A}(X, h, i, \chi)$ if the classification and the indexing map, h, satisfy the slowly varying condition

$$\chi(x) - 1 \leq \chi \circ h(x) \leq \chi(x) + 1$$

for all $x \in X$. Given a positive sequence, (α_n), an (α_n)-***classified pseudo-orbit*** or simply an (α_n)-***pseudo orbit*** is a classified pseudo-orbit of M and f for which for all $x \in X$, we have $d(i \circ h(x), f \circ i(x)) < \alpha(h(x)) = \alpha_{\chi \circ h(x)}$. A classified pseudo-orbit is *closed* if each of the sets $i(P_n)$ is a closed subset of M. A C^0 (C^r) (α_n)-pseudo orbit of M is an (α_n)-pseudo orbit of M which is C^0 (C^r).

The *inverse classified pseudo-orbit* of M and f^{-1} associated to $\mathfrak{A}(X, h, i, \chi)$ is $\mathfrak{A}^{-1}(X, h^{-1}, i, \chi)$

We can use the minimal classification of X to define weaker classes of continuity. A classified pseudo-orbit, $\mathfrak{A}(X, h, i, \chi)$, is C^0_* if for a given metric of X and for each $0 \leq n$ the functions $h\big|_{P_n}$ and $i\big|_{P_n}$ are C^0. For $1 \leq \kappa$ and $1 \leq r$, a pseudo-orbit, $\mathfrak{A}$, is C^r_* if $\mathfrak{A}$ is a C^0_* pseudo-orbit for which there exists a C^r manifold $\bar{X}$, a collection of open subsets, $\bar{P}_n \subset \bar{X}$, a C^r diffeomorphism $\bar{h} : \bar{X} \to \bar{X}$, and a C^r map $\bar{i} : \bar{X} \to M$ for which $X \subset \bar{X}$, $P_n \subset \bar{P}_n$, $h = \bar{h}\big|_X$, $i = \bar{i}\big|_X$, and moreover, there exists a positive constant K for which for each $0 \leq n$ the C^r norms of $\bar{h}\big|_{\bar{P}_n}$ and $\bar{i}\big|_{\bar{P}_n}$ are bounded by $K\kappa^n$.

Given a pair of classified pseudo-orbits, $\mathfrak{A}(X, h, i, \chi)$ and $\mathfrak{B}\left(\bar{X}, \bar{h}, \bar{i}, \bar{\chi}\right)$, the pseudo-orbit $\mathfrak{B}$ is *monotonically embedded* in $\mathfrak{A}$ if $\mathfrak{B}$ is embedded in $\mathfrak{A}$ and $\chi \circ k_e(x) \leq \bar{\chi}(x)$ for all $x \in \bar{X}$. Similarly, the pseudo-orbit $\mathfrak{B}$ is a *monotonic factor* of $\mathfrak{A}$ if $\mathfrak{B}$ is a factor of $\mathfrak{A}$ and $\bar{\chi} \circ k_f(x) \leq \chi(x)$ for all $x \in X$.

Given a a classified pseudo-orbit $\mathfrak{A}$ of M and f and a positive sequence (R_n), we define the (R_n)-*perturbation constant* of $\mathfrak{A}$ to be $\mathfrak{p}_{(R_n)}(\mathfrak{A}) = \sup_{x \in X} \frac{d(i(h(x)), f(i(x)))}{R(\chi(x))}$.

For a classified pseudo-orbit $\mathfrak{A}(X, h, i, \chi)$ of M and f and a sequence (β_n), the pseudo-orbit is (β_n)-*shadowed* by a particular invariant set $\mathfrak{B}\left(X, h, \bar{i}, \chi\right)$, if $d\left(i(x), \bar{i}(x)\right) \leq \beta(x) = \beta_{\chi(x)}$ for all $x \in X$.

8.3 The Classifying manifold, M

For uniformly hyperbolic invariant sets, the proof of the Stable manifold theorem is greatly simplified by working in a adapted metric. This adapted metric is chosen so that the eventual contraction conditions of uniform hyperbolicity are changed into immediate contraction.

Similarly, for weakly hyperbolic invariant sets, the proof of the Stable manifold theorem is also simplified by working in an adapted metric with similar properties (see Theorem 10.1 below). Unfortunately, for a weakly hyperbolic invariant set, the "adapted metric" which is required is actually a *countable family* of adapted C^∞ Riemannian metrics of M. In order to form *one* metric out of this family of metrics, it will be convenient to work with a disjoint union of a countable collection of copies of M. We call the resulting Riemannian manifold, $\boldsymbol{M}$, the classifying manifold of M.

More importantly, the conditions of weak hyperbolicity, to be discussed in the next chapter, rather naturally suggest the the division of a weakly hyperbolic invariant set into a countable collection of (non-invariant) "hyperbolicity classes" or "hyperbolic blocks". While, for instance, the stable and unstable manifolds of a weakly hyperbolic invariant set are *not* continuous over the whole invariant set, they *are* continuous over each "hyperbolic block".

Not surprisingly, exactly one adapted metric in the countable collection of adapted metrics is associated to each hyperbolic block. All of the constructions in the remainder of this thesis will ensure that each hyperbolic block is naturally associated with exactly one copy of M in the classifying manifold $\boldsymbol{M}$.

We define the *classifying manifold* of M, denoted by the symbol $\boldsymbol{M}$, to be $\boldsymbol{M} = \bigsqcup_0^\infty M$ (disjoint union). Furthermore, let M_n denote the n^{th} copy of M in $\boldsymbol{M}$. Note that $\boldsymbol{M}$ is isomorphic to $\mathbb{Z}^+ \times M$. This means that there is a pair of projections projecting onto the first, π_b, and second, π_M, components of the cartesian product. Note that both π_b and π_M make $\boldsymbol{M}$ into bundles over $\mathbb{Z}^+$ and M respectively.

Since M is a compact Riemannian manifold, $\boldsymbol{M}$ is σ-compact and hence is a paracompact Riemannian manifold. Since all of the disjoint copies of M in $\boldsymbol{M}$ are really the same manifold, we can, trivially, identify any two copies, M_n and M_m. We will often need to make such an identification explicit. For $0 \leq n, m$, we use the symbols $Id_{(m,n)}$ to denote this standard identification of M_n with M_m, i.e. $Id_{(m,n)}$ is the identification map $Id_{(m,n)} : M_n \to M_m$. We use the symbols $Id_{(*,n)}$ and $Id_{(n,*)}$ to denote the standard identifications $Id_{(*,n)} : M_n \to M$ and $Id_{(n,*)} : M \to M_n$.

There are essentially two different C^∞ Riemannian metrics which we might place on $\boldsymbol{M}$. The first is the adapted metric noted above. We will often use the symbol $\boldsymbol{M}_a$ to denote $\boldsymbol{M}$ together with an adapted metric. We can also consider the Riemannian metric of $\boldsymbol{M}$ induced by using the original metric on all of the copies of M in $\boldsymbol{M}$. We will often use the symbol $\boldsymbol{M}_o$ to denote this metric.

The classifying manifold, $\boldsymbol{M}$, together with a specific metric on $\boldsymbol{M}$, is (A_n)-

(B_n)-*topologically equivalent* to $\boldsymbol{M}_o$, if for each $0 \leq n$, the manifold M_n with the given metric is A_n-B_n-topologically equivalent to the manifold M_n with the original metric of M.

In analogy with our previous definitions of κ-slowly varying functions, we can define a sequence, (B_n), to be κ-*slowly varying* if

$$\frac{1}{\kappa} \leq \frac{B_{n+1}}{B_n} \leq \kappa.$$

A κ-*slowly increasing (decreasing) sequence* is a κ-slowly varying sequence which monotonically increases (decreases) as $n \to \infty$. Among other facts, Theorem 10.1 proves that there exists a κ^2 slowly varying sequence, (B_n), for which $\boldsymbol{M}_*$ is $\frac{1}{\sqrt{\lambda}}$-(B_n)-topologically equivalent to $\boldsymbol{M}_o$.

8.4 Pseudo-orbits and the Classifying manifold

Having defined the classifying manifold $\boldsymbol{M}$, it is natural to consider pseudo-orbits of $\boldsymbol{M}$. In order to distinguish between pseudo-orbits of M and $\boldsymbol{M}$, we will generally use capital Gothic characters $\mathfrak{A}$, or $\mathfrak{B}$ to denote pseudo-orbits of M, and bold Gothic characters $\boldsymbol{\mathfrak{A}}$, or $\boldsymbol{\mathfrak{B}}$ to denote pseudo-orbits of $\boldsymbol{M}$. In the following discussion of pseudo-orbits of $\boldsymbol{M}$ we are intentionally vague about which metric we place on $\boldsymbol{M}$. We will make use of pseudo-orbits of any of $\boldsymbol{M}_*$, and $\boldsymbol{M}_o$ at various times in the rest of this thesis.

As we noted in the introduction to this chapter we are primarily interested in the definition of a *factored* pseudo-orbit of $\boldsymbol{M}$. Before we can make this particular definition we must first define the concept of an unfactored pseudo-orbit of $\boldsymbol{M}$. An *(unfactored classified)* (α_n)-*pseudo orbit*, $\boldsymbol{\mathfrak{A}}$, for the manifold $\boldsymbol{M}$ and the diffeomorphism f, is a quintuple $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{\imath}, \chi, \boldsymbol{f})$, where

1. $\boldsymbol{X}$ is the disjoint union of a countable collection of sets $\boldsymbol{X}_n$,

2. χ is the classifying map from $\boldsymbol{X}$ to $\mathbb{Z}^+$ defined by $\chi(x) = n$ for all $x \in \boldsymbol{X}_n$,

3. $\boldsymbol{h}$ is a bijection of $\boldsymbol{X}$ which, for all $x \in \boldsymbol{X}$, satisfies the slowly varying condition,

$$\chi(x) - 1 \leq \chi \circ \boldsymbol{h}(x) \leq \chi(x) + 1$$

4. $\boldsymbol{i}$ is a map from $\boldsymbol{X}$ into $\boldsymbol{M}$ which respects the classifying map, χ, that is $\pi_b \circ \boldsymbol{i} = \chi$,

5. $\boldsymbol{f}$ is the function from $\boldsymbol{i}(\boldsymbol{X})$ to $\boldsymbol{M}$ for which if we denote $x \in \boldsymbol{X}_n$ by (n, x) then $\boldsymbol{f}$ is defined by

$$\boldsymbol{f}(\boldsymbol{i}(j,x)) = (\chi \circ \boldsymbol{h}(j,x), f\pi_m \boldsymbol{i}(j,x)),$$

and moreover $\boldsymbol{X}$, $\boldsymbol{h}$, $\boldsymbol{i}$, and $\boldsymbol{f}$ make the first and second squares of the following diagram (α_n)-commute and commute respectively.

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{i}"] \arrow[d, "\boldsymbol{h}"'] & \boldsymbol{i}(\boldsymbol{X}) \arrow[r, "\pi_M"] \arrow[d, "\boldsymbol{f}"'] & \pi_M(\boldsymbol{i}(\boldsymbol{X})) \arrow[d, "f"'] \\
\boldsymbol{X} \arrow[r, "\boldsymbol{i}"] & \boldsymbol{M} \arrow[r, "\pi_M"] & M
\end{tikzcd}$$

The fact that the first square (α_n)-commutes means that for all $x \in \boldsymbol{X}_n$, we have that $d_n(\boldsymbol{i} \circ \boldsymbol{h}(x), \boldsymbol{f} \circ \boldsymbol{i}(x)) < \alpha_n$ where $d_n(\cdot,\cdot)$ denotes the distance metric of M_n. As above, we call $\boldsymbol{X}$, $\boldsymbol{i}$, and $\boldsymbol{h}$ the index space, map, and bijection respectively. Note that while the map i for a pseudo-orbit of M and f is required to be injective, the map $\boldsymbol{i}$ for a pseudo-orbit of $\boldsymbol{M}$ and f need not be 1-1. Again, an unfactored pseudo-orbit is *closed* if each of the sets, $\boldsymbol{i}(\boldsymbol{P}_n)$, is a closed subset of $\boldsymbol{M}$.

We call χ the classifying map. We call the collection, $\{\boldsymbol{X}_n\}_0^\infty$, the partition of $\boldsymbol{X}$. Again, as above we can define the gradation of $\boldsymbol{X}$ to be the collection of sets, $\{\boldsymbol{P}_n\}_0^\infty$, defined by $\boldsymbol{P}_n = \bigcup_{i=0}^n \boldsymbol{X}_n$. Note that for $0 \leq n < m$, $\boldsymbol{X}_n \bigcap \boldsymbol{X}_m = \emptyset$, and $\boldsymbol{P}_n \subset \boldsymbol{P}_m$. From this definition we know that $\boldsymbol{X} = \bigcup_{n=0}^\infty \boldsymbol{P}_n$. Given a sequence (R_n), and this classification of $\boldsymbol{X}$, then we can define a function $R : \boldsymbol{X} \to \mathbf{R}$ by $R(x) = R_{\chi(x)}$.

A classified pseudo-orbit, $\mathfrak{A}(X, h, i, \chi)$, of M and f, is a *(monotonic) factor* of an unfactored classified (α_n)-pseudo orbit, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \chi, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, if there exists a surjective map k from $\boldsymbol{X}$ to X for which for all $x \in \boldsymbol{X}$, $\chi \circ k(x) \leq \chi(x)$, and for which the following diagram commutes

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{h}"] \arrow[d, "k"'] & \boldsymbol{X} \arrow[r, "\boldsymbol{i}"] \arrow[d, "k"'] & \boldsymbol{M} \arrow[d, "\pi_M"'] \\
X \arrow[r, "h"] & X \arrow[r, "i"] & M
\end{tikzcd}$$

The surjective maps, k and π_M, make $\boldsymbol{X}$ and $\boldsymbol{M}$ into a bundles over X and M respectively. With this interpretation, the above diagram states that the maps, $\boldsymbol{h}$ and $\boldsymbol{i}$, are bundle maps between the appropriate bundles.

A *(factored) (classified)* (α_n)*-pseudo orbit of* $\boldsymbol{M}$ *and* f is a classified pseudo-orbit, $\mathfrak{A}(X, h, i, \chi)$, of M and f, which is a monotonic factor of an unfactored classified (α_n)-pseudo orbit, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, which is, moreover, factored by the monotonic factor map, $k : \boldsymbol{X} \to X$. We denote such a factored (α_n)-pseudo orbit of $\boldsymbol{M}$ and f by the symbol $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$. A factored classified pseudo-orbit is *closed* if both the factor pseudo-orbit (of M and f) as well as the unfactored pseudo-orbit (of $\boldsymbol{M}$ and f) are both closed. Note that to specify a (factored) pseudo-orbit of $\boldsymbol{M}$ and f we need only specify a classified pseudo-orbit of M and f as well as the monotonic factor map, k, the classified index space $\boldsymbol{X}$ and the classified bijection $\boldsymbol{h}$.

The factored pseudo-orbit, $\boldsymbol{\mathfrak{A}}$, of $\boldsymbol{M}$ and f is called a *lift* of the classified pseudo-orbit, $\mathfrak{A}$, of M and f. The pseudo-orbit $\boldsymbol{\mathfrak{A}}$ is a *minimal factored pseudo-orbit* if $\chi \circ k = \boldsymbol{\chi}$. The pseudo-orbit $\boldsymbol{\mathfrak{A}}$ is a *minimal lift* of the pseudo-orbit $\mathfrak{A}$ if $\boldsymbol{\mathfrak{A}}$ is a minimal factored pseudo-orbit *and* k is a bijection of $\boldsymbol{X}$ and X.

Just as for pseudo-orbits of M and f, we can define the *inverse factored pseudo-orbit*, $\boldsymbol{\mathfrak{A}}^{-1}\left(\mathfrak{A}^{-1}, k, \boldsymbol{X}, \boldsymbol{h}^{-1}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f}^{-1}\right)$, and the *inverse unfactored pseudo-orbit*, $\boldsymbol{\mathfrak{A}}^{-1}\left(\boldsymbol{X}, \boldsymbol{h}^{-1}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f}^{-1}\right)$, which respectively correspond to any factored, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, or unfactored, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$ pseudo-orbit of $\boldsymbol{M}$ and f.

Unless stated otherwise, *all* of the pseudo-orbits of $\boldsymbol{M}$ and f which will be used in this thesis will be *factored* (α_n)-pseudo orbits of $\boldsymbol{M}$ and f for some sequence (α_n).

For any (factored) pseudo-orbit, the following cubic diagram $((\alpha_n) -)$ commutes

$$
\begin{tikzcd}
 & \boldsymbol{i}(\boldsymbol{X}) \arrow[rr, "\boldsymbol{f}"] \arrow[dd, "\pi_M"' near start] & & \boldsymbol{M} \arrow[dd, "\pi_M"] \\
\boldsymbol{X} \arrow[ur, "\boldsymbol{i}"] \arrow[rr, "\boldsymbol{h}" near end] \arrow[dd, "k"'] & & \boldsymbol{X} \arrow[ur, "\boldsymbol{i}"'] \arrow[dd, "k" near start] & \\
 & M \arrow[rr, "f" near start] & & M \\
X \arrow[ur, "i"] \arrow[rr, "h"] & & X \arrow[ur, "i"'] &
\end{tikzcd}
$$

That is, all of the squares commute in the diagram above *except* for the top and bottom squares which both (α_n)-commute.

Note that any ***unfactored invariant set*** of $\boldsymbol{M}$ and f is automatically a *factored* invariant set of $\boldsymbol{M}$ and f. To see this, corresponding to any unfactored invariant set, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, we must construct a classified invariant set, $\mathfrak{A}(X, h, i, \chi)$, of M and f and a monotonic factor map $k : \boldsymbol{X} \to X$. Let $k = \pi_M \circ \boldsymbol{i}$, $X = k(\boldsymbol{X})$, $h = f\big|_X$, $i = Id_M\big|_X$, and $\chi(x) = \inf_{y \in k^{-1}(x)} \{\boldsymbol{\chi}(y)\}$. Since $\boldsymbol{\mathfrak{A}}$ is an invariant set of $\boldsymbol{M}$ and f it is easy to see that $\mathfrak{A}(X, h, i)$ is an invariant set of M and f. Moreover, if χ is a classifying map for $\mathfrak{A}$, then it is also easy to see that, by construction, k is a monotonic factor map from $\boldsymbol{\mathfrak{A}}$ to $\mathfrak{A}$. The only important thing to show is then that χ is indeed a classifying map for $\mathfrak{A}$. Since $\boldsymbol{\chi}$ can only take on integral values we know that for every x in X, there exists a y in $k^{-1}(x)$ for which $\chi(x) = \boldsymbol{\chi}(y)$. Given x in X and let y in $k^{-1}(x)$ be such that $\chi(x) = \boldsymbol{\chi}(y)$, then we know that

$$\chi(x) - 1 \leq \boldsymbol{\chi}(y) - 1 \leq \boldsymbol{\chi} \circ \boldsymbol{h}(y) = \chi \circ h \circ k(y).$$

A similar argument shows that $\chi \circ h(x) \leq \chi(x) + 1$. This pair of inequalities then proves that χ is a classifying map for $\mathfrak{A}$ and hence that $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$ is a *factored* invariant set for $\boldsymbol{M}$ and f.

Since $\boldsymbol{M}$ is not compact we must also be more careful about the continuity properties of pseudo-orbits of $\boldsymbol{M}$. In particular we will not assume that the functions, $\boldsymbol{h}$ and $\boldsymbol{i}$, are continuous on the *whole* of $\boldsymbol{X}$. In order to discuss the continuity properties of a pseudo-orbit we define

$$\boldsymbol{X}_{n-} = \boldsymbol{X}_n \bigcap \boldsymbol{h}^{-1}(\boldsymbol{X}_{n-1}).$$

$$
\begin{aligned}
\boldsymbol{X}_{n\bullet} &= \boldsymbol{X}_n \bigcap \boldsymbol{h}^{-1}(\boldsymbol{X}_n), \\
\boldsymbol{X}_{n+} &= \boldsymbol{X}_n \bigcap \boldsymbol{h}^{-1}(\boldsymbol{X}_{n+1}), \\
\boldsymbol{X}_{n-}^{(-1)} &= \boldsymbol{X}_n \bigcap \boldsymbol{h}(\boldsymbol{X}_{n-1}), \\
\boldsymbol{X}_{n\bullet}^{(-1)} &= \boldsymbol{X}_n \bigcap \boldsymbol{h}(\boldsymbol{X}_n), \quad \text{and} \\
\boldsymbol{X}_{n+}^{(-1)} &= \boldsymbol{X}_n \bigcap \boldsymbol{h}(\boldsymbol{X}_{n+1}).
\end{aligned}
$$

A C^0_* (α_n)*-pseudo orbit of* $\boldsymbol{M}$, is an (α_n)-pseudo orbit of $\boldsymbol{M}$ for which the factor pseudo-orbit, $\mathfrak{A}$, is a C^0_* pseudo-orbit of M and f, the set $\boldsymbol{X}$ is a topological space and for all $0 \leq n$ the indexing map, $\boldsymbol{i}$, is continuous on each $\boldsymbol{X}_n$, the indexing map, $\boldsymbol{h}$, is continuous on the sets $\boldsymbol{X}_{n-}$, $\boldsymbol{X}_{n\bullet}$, and $\boldsymbol{X}_{n+}$, and the indexing map, $\boldsymbol{h}^{-1}$, is continuous on the sets $\boldsymbol{X}_{n-}^{(-1)}$, $\boldsymbol{X}_{n\bullet}^{(-1)}$, and $\boldsymbol{X}_{n+}^{(-1)}$.

Given two pseudo-orbits, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \chi, \boldsymbol{f})$ and $\boldsymbol{\mathfrak{B}}\left(\mathfrak{A}, \tilde{k}, \tilde{\boldsymbol{X}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{i}}, \tilde{\chi}, \tilde{\boldsymbol{f}}\right)$, of $\boldsymbol{M}$ and f, $\boldsymbol{\mathfrak{B}}$ is a ***sub-pseudo-orbit*** of $\boldsymbol{\mathfrak{A}}$, if $\mathfrak{B}$ is a sub-pseudo orbit of $\mathfrak{A}$, and if $\tilde{\boldsymbol{X}} \subset \boldsymbol{X}$, $\tilde{k} = k\big|_{\tilde{\boldsymbol{X}}}$, $\tilde{\boldsymbol{h}} = \boldsymbol{h}\big|_{\tilde{\boldsymbol{X}}}$, and $\tilde{\boldsymbol{i}} = \boldsymbol{i}\big|_{\tilde{\boldsymbol{X}}}$. Note that since $\boldsymbol{\mathfrak{B}}$ is itself a pseudo-orbit, the set $\tilde{\boldsymbol{X}}$ is $\boldsymbol{h}$-invariant.

Again, more generally, the pseudo-orbit $\boldsymbol{\mathfrak{B}}$ is *embedded* in the pseudo-orbit $\boldsymbol{\mathfrak{A}}$ if $\mathfrak{B}$ is embedded in $\mathfrak{A}$ via the injective function $k_e : \tilde{X} \to X$ and if there exists an injective function $\boldsymbol{k}_e : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$ which makes the following diagrams commute

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{h}"] & \boldsymbol{X} \arrow[dr, "\boldsymbol{i}"] & \\
& & \boldsymbol{M} \\
\tilde{\boldsymbol{X}} \arrow[uu, "\boldsymbol{k}_e"] \arrow[r, "\tilde{\boldsymbol{h}}"'] & \tilde{\boldsymbol{X}} \arrow[uu, "\boldsymbol{k}_e"'] \arrow[ur, "\tilde{\boldsymbol{i}}"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[d, "k"'] & \tilde{\boldsymbol{X}} \arrow[l, "\boldsymbol{k}_e"'] \arrow[d, "\tilde{k}"] \\
X & \tilde{X} \arrow[l, "k_e"]
\end{tikzcd}
$$

The pseudo-orbit $\mathfrak{B}$ is *monotonically embedded* in $\mathfrak{A}$ if $\mathfrak{B}$ is embedded in $\mathfrak{A}$, $\mathfrak{B}$ is monotonically embedded in $\mathfrak{A}$, and $\chi \circ \boldsymbol{k}_e(x) \leq \tilde{\chi}(x)$ for all $x \in \tilde{\boldsymbol{X}}$.

Dually, the pseudo-orbit $\mathfrak{B}$ is a *factor* of the pseudo-orbit $\mathfrak{A}$ if $\mathfrak{B}$ is a factor of $\mathfrak{A}$ via the surjective function $k_f : X \to \tilde{X}$ and if there exists an surjective function $\boldsymbol{k}_f : \boldsymbol{X} \to \tilde{\boldsymbol{X}}$ which makes the following diagrams commute

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{h}"] \arrow[d, "\boldsymbol{k}_f"'] & \boldsymbol{X} \arrow[d, "\boldsymbol{k}_f"'] \arrow[dr, "\boldsymbol{i}"] & \\
& & \boldsymbol{M} \\
\tilde{\boldsymbol{X}} \arrow[r, "\tilde{\boldsymbol{h}}"] & \tilde{\boldsymbol{X}} \arrow[ur, "\tilde{\boldsymbol{i}}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "k_f"] \arrow[d, "k"'] & \tilde{\boldsymbol{X}} \arrow[d, "\tilde{k}"] \\
X \arrow[r, "k_f"] & \tilde{X}
\end{tikzcd}$$

The pseudo-orbits, $\mathfrak{A}$ and $\mathfrak{B}$, are *conjugate* if k_f is a homeomorphism. The pseudo-orbit $\mathfrak{B}$ is a *monotonic factor* of $\mathfrak{A}$ if $\mathfrak{B}$ is a factor of $\mathfrak{A}$, $\mathfrak{B}$ is a monotonic factor of $\mathfrak{A}$, and $\tilde{\chi} \circ \boldsymbol{k}_f(x) \leq \chi(x)$ for all $x \in \tilde{\boldsymbol{X}}$.

Just as for pseudo-orbits of M, the perturbation constant and the concept of shadowing are important for pseudo-orbits of $\boldsymbol{M}$. Since $\boldsymbol{X}$ has the natural minimal classification, χ, we can define the (R_n)-perturbation constant and the concept of (β_n)-shadowing to be analogous to the non-uniform definitions given above. More precisely, for a sequence (R_n), we define the (R_n)-*perturbation constant*, $\mathfrak{p}_{(R_n)}(\mathfrak{A})$, of a pseudo-orbit, $\mathfrak{A}$, of $\boldsymbol{M}$ to be $\mathfrak{p}_{(R_n)}(\mathfrak{A}) = \sup_{x \in \boldsymbol{X}} \frac{d(\boldsymbol{i}(\boldsymbol{h}(x)), \boldsymbol{f}(\boldsymbol{i}(x)))}{R(x)}$. Similarly, for a sequence, (β_n), an invariant set $\mathfrak{B}\left(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{i}}, \tilde{\boldsymbol{f}}\right)$ of $\boldsymbol{M}$ (β_n)-*shadows* a pseudo-orbit $\mathfrak{A}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{f})$ of $\boldsymbol{M}$ if for all $x \in \boldsymbol{X}$, $d\left(\boldsymbol{i}(x), \tilde{\boldsymbol{i}}(x)\right) \leq \beta(x)$.

Any α-pseudo orbit of M can be trivially identified with an α-pseudo orbit of $\boldsymbol{M}$ by identifying M with M_n, for some n. Conversely any pseudo-orbit of $\boldsymbol{M}$ for which $\boldsymbol{i}(\boldsymbol{X})$ is contained in $\boldsymbol{M}_n$ for some n can be similarly trivially identified with an unclassified pseudo-orbit of M. We can, by making use of these identifications, extend the concept of factors between pairs of pseudo-orbits of $\boldsymbol{M}$ to factors between a uniform pseudo-orbit of M and a pseudo-orbit of $\boldsymbol{M}$.

An *n-strong* pseudo-orbit of $\boldsymbol{M}$ is a pseudo-orbit of $\boldsymbol{M}$ for which $\boldsymbol{i}(\boldsymbol{X})$ is contained in $\bigsqcup_{i=0}^{n} M_i$. A *strong* pseudo-orbit of $\boldsymbol{M}$ is a pseudo-orbit of $\boldsymbol{M}$ which is n-strong for some $0 \leq n$. An *n-superstrong* pseudo-orbit of $\boldsymbol{M}$ is a pseudo-orbit of $\boldsymbol{M}$ for which $\boldsymbol{i}(\boldsymbol{X})$ is contained in M_n. From above, an n-superstrong pseudo-orbit of $\boldsymbol{M}$ can be trivially identified with a uniform pseudo-orbit of M.

Recall that an (α_n)-pseudo orbit, $\mathfrak{A}(X, h, i, \chi)$ of M and f is a classified pseudo-orbit of M for which the indexing set X has an associated minimal classification, χ. We can trivially identify the (α_n)-pseudo orbit $\mathfrak{A}(X, h, i, \chi)$ of M and f with the (α_n)-pseudo orbit, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$ of $\boldsymbol{M}_o$ and f where

$$
\begin{aligned}
\boldsymbol{X} &= \bigsqcup_{n=0}^{\infty} X_n, \\
k(j, x) &= x, \\
\boldsymbol{\chi}(j, x) &= \chi(x), \\
\boldsymbol{h}(j, x) &= (\chi \circ h(x), h(x)), \\
\boldsymbol{i}(j, x) &= (\chi(x), i(x)).
\end{aligned}
$$

Note that this $\boldsymbol{\mathfrak{A}}$, that we have just constructed, is the minimal lift of $\mathfrak{A}$. Conversely, any factored (α_n)-pseudo orbit of $\boldsymbol{M}_1$ can be *trivially* identified with an (α_n)-pseudo orbit of M and f.

Given a pseudo-orbit, $\boldsymbol{\mathfrak{A}}$ and a point $x \in \boldsymbol{X}$ the orbit $\left\{\boldsymbol{h}^i(x)\right\}_{-\infty}^{\infty}$ is *maximally weak with base* x if $x \in \boldsymbol{X}_n$ implies that $\boldsymbol{h}^m(x) \in \boldsymbol{X}_{n+|m|}$ for all $m \in \mathbb{Z}$. For a given pseudo-orbit, $\boldsymbol{\mathfrak{A}}$, the orbit $\left\{\boldsymbol{h}^i(y)\right\}_{-\infty}^{\infty}$ is an *n-shift* of the orbit $\left\{\boldsymbol{h}^i(x)\right\}_{-\infty}^{\infty}$ if for all $i \in \mathbb{Z}$, $\boldsymbol{h}^i(x) \in \boldsymbol{X}_j$ implies that $\boldsymbol{h}^i(y) \in \boldsymbol{X}_{j+n}$ and if $\boldsymbol{i} \circ \boldsymbol{h}^i(y) = Id_{(\chi(x), \chi(x)+n)} \circ \boldsymbol{i} \circ \boldsymbol{h}^i(x)$. A pseudo-orbit of $\boldsymbol{M}$ and f is *maximally shifted* if

1. for every $x \in \boldsymbol{X}$ there exists an $n \in \mathbb{Z}$ for which the orbit $\left\{\boldsymbol{h}^i(x)\right\}_{-\infty}^{\infty}$ is maximally weak with base $\boldsymbol{h}^n(x)$,

2. for every $x \in \boldsymbol{X}$ and every $0 \leq n$ the there exists a $y \in \boldsymbol{X}$ for which the orbit $\left\{\boldsymbol{h}^i(y)\right\}_{-\infty}^{\infty}$ is an n-shift of the orbit $\left\{\boldsymbol{h}^i(x)\right\}_{-\infty}^{\infty}$.

Given a pseudo-orbit, $\mathfrak{A}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \chi, \boldsymbol{f})$, we can construct the *maximally shifted closure* of $\mathfrak{A}$, which we denote by the symbol $\overline{\mathfrak{A}}\left(\overline{\mathfrak{A}}, \overline{k}, \overline{\boldsymbol{X}}, \overline{\boldsymbol{h}}, \overline{\boldsymbol{i}}, \overline{\chi}, \overline{\boldsymbol{f}}\right)$, as follows. Let $\overline{\boldsymbol{X}} = \mathbb{Z} \times \mathbb{Z}^{+} \times \boldsymbol{X}$, where $\overline{\boldsymbol{X}}_n = \{n\} \times \mathbb{Z}^{+} \times \boldsymbol{X}$. For (m, n, x) in $\overline{\boldsymbol{X}}$ define

$$
\begin{aligned}
\overline{\boldsymbol{h}}(m,n,x) &= (m+1, n, h(x)), \\
\overline{\chi}(m,n,x) &= \chi(x) + |m| + n, \\
\overline{\boldsymbol{i}}(m,n,x) &= Id_{(\overline{\chi}(m,n,x), \chi\circ\boldsymbol{h}^m(x))} \circ \boldsymbol{i} \circ h^m(x), \quad \text{and} \\
\overline{k}(m,n,x) &= k(x).
\end{aligned}
$$

In particular, note that, since χ is a classifying function for $\mathfrak{A}$, we know that $\chi\circ\boldsymbol{h}^m(x) \leq \overline{\chi}(m,n,x)$ for all (m,n,x) in $\overline{\boldsymbol{X}}$. We will also need to define the maps

$$
\begin{aligned}
\overline{k}_{\boldsymbol{X}} : \boldsymbol{X} \to \overline{\boldsymbol{X}}, &\quad \text{and} \quad \overline{k}_f : \overline{\boldsymbol{X}} \to \boldsymbol{X} \ \text{ by} \\
\overline{k}_{\boldsymbol{X}}(x) = (0,0,x), &\quad \text{and} \quad \overline{k}_f(m,n,x) = x.
\end{aligned}
$$

With these definitions, it is easy to see that

1. $\overline{\mathfrak{A}}$ is a pseudo-orbit of $\boldsymbol{M}$ and f,

2. $\overline{\mathfrak{A}}$ is a maximally shifted pseudo-orbit,

3. $\overline{\chi} \circ \overline{k}_{\boldsymbol{X}} = \chi$,

4. for every $x \in \boldsymbol{X}$, the orbit $\left\{\overline{\boldsymbol{h}}^n \circ \overline{k}_{\boldsymbol{X}}(x)\right\}_{-\infty}^{\infty}$ is maximally weak with base $\overline{k}_{\boldsymbol{X}}(x)$,

5. for every $\overline{x} \in \overline{\boldsymbol{X}}$ there exists an $x \in \boldsymbol{X}$ and an $0 \leq n$ for which the orbit $\left\{\overline{\boldsymbol{h}}^m(\overline{x})\right\}_{-\infty}^{\infty}$ is an n-shift of the orbit $\left\{\overline{\boldsymbol{h}}^m \circ \overline{k}_{\boldsymbol{X}}(x)\right\}_{-\infty}^{\infty}$,

6. $\mathfrak{A}$ is a monotonic factor of $\overline{\mathfrak{A}}$ via the monotonic factor map $\overline{\boldsymbol{k}}_f$,

7. $\mathfrak{A}$ is a monotonic factor of $\overline{\mathfrak{A}}$ via the monotonic factor map $k \circ \overline{\boldsymbol{k}}_f$, and

8. $\mathfrak{A}$ is an invariant set iff $\overline{\mathfrak{A}}$ is an invariant set.

If $\mathfrak{A}$ is C^0_* then we can use the metric of $\boldsymbol{X}$ to define the following metric of $\overline{\boldsymbol{X}}$. For (m,n,x) and $(\hat{m}, \hat{n}, \hat{x})$ in $\overline{\boldsymbol{X}}$ we define

$$
d((m,n,x),(\hat{m},\hat{n},\hat{x})) = \begin{cases} \infty & \text{if } \ m \neq \hat{m} \ \text{ or } \ n \neq \hat{n}, \\ d(x,\hat{x}) & \text{if } \ m = \hat{m} \ \text{ and } \ n = \hat{n}. \end{cases}
$$

Note that this implies that the sequence, $\{(m_i, n_i, x_i)\}_0^\infty$, converges iff the sequence $\{x_i\}_0^\infty$ converges and there exists $0 < I, M \in \mathbf{Z}$, and $0 \leq N$ such that for all $I \leq i$, $m_i = M$, and $n_i = N$. With this metric on $\boldsymbol{X}$ it is then easy to see that $\mathfrak{X}$ is C^0_* and moreover, that $\bar{k}_X$ is Lipschitz. To see that $\mathfrak{X}$ is C^0_*, note that $\boldsymbol{X}_{n*}$, $\boldsymbol{X}_{n+}$, $\boldsymbol{X}^{(-1)}_{n-}$, and $\boldsymbol{X}^{(-1)}_{n*}$ are all empty and that $\boldsymbol{X}_{n-} = \boldsymbol{X}^{(-1)}_{n+} = \boldsymbol{X}_n$.

Chapter 9

Pseudo-Hyperbolicity of Pseudo-Orbits

Recall that the Weak Shadowing Stable Manifold theorem is stated in terms of (weakly) pseudo-hyperbolic pseudo-orbits. The previous chapter dealt with the definition and properties of pseudo-orbits. The current chapter will cover the definition of (pseudo) hyperbolicity of a pseudo-orbit.

9.1 Hyperbolicity of invariant sets

Uniformly hyperbolic and weakly hyperbolic invariant sets are examples of pseudo-hyperbolic pseudo-orbits which are, moreover, well "understood" by dynamical systems theory. As such, we quickly review the definitions of these simpler objects in order that the reader can more easily understand the definitions of the more complex instances.

For the rest of this chapter we again assume that we are given a single, fixed, compact Riemannian manifold, M, together with a single, fixed, $C^{r+\gamma}$ diffeomorphism, f, which maps M to itself. Recall that a subset, Λ, of M is an f-invariant set if there exists a 0-pseudo orbit, $\Lambda(X, h, i)$, for which $i(X) = \Lambda$. Since i is injective this essentially corresponds to the "classical" definition of an invariant set: $f^{-1}(\Lambda) = \Lambda = f(\Lambda)$.

For a given invariant set $\Lambda(X, h, i)$, as part of the definition of the hyperbolicity of the set Λ, we require that there exists a Tf-invariant splitting of the tangent space $T_\Lambda M$ of the set Λ, in the manifold M. More specifically we ask that there exist two subbundles $\boldsymbol{E}^s_\Lambda$ and $\boldsymbol{E}^u_\Lambda$ of $T_\Lambda M$, such that $T_\Lambda M = \boldsymbol{E}^s_\Lambda \oplus_\Lambda \boldsymbol{E}^u_\Lambda$. This sum is the Whitney direct sum of vector bundles over Λ. In particular this means that the fibres of $T_\Lambda M$ are the direct sums of the fibres of $\boldsymbol{E}^s_\Lambda$ and $\boldsymbol{E}^u_\Lambda$, that is $T_x M = \boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x$ for all $x \in \Lambda$.

Associated with the splitting in each fibre is a pair of projection operators which project vectors of the fibre *onto* one of the two subspaces of the fibre *along* cosets of the other. We will denote the projection onto the stable, $\boldsymbol{E}^s_x$, and unstable $\boldsymbol{E}^u_x$, subspaces by the symbols p^s_x and p^u_x respectively.

We can then define an invariant set, $\Lambda(X, h, i)$, to be *weakly λ-hyperbolic* for $0 < \lambda < 1$ if there exists an Tf-invariant splitting $T_\Lambda M = \boldsymbol{E}^s_\Lambda \oplus_\Lambda \boldsymbol{E}^\Lambda_u$, and a function $C_\lambda : X \to [1, \infty)$ for which

$$
\begin{aligned}
|Df^n(v_s)|_{h^n(x)} &\leq C_\lambda(x)\,\lambda^n\,|v_s|_x && (9.1\text{a})\\
\left|Df^{-n}(v_u)\right|_{h^{-n}(x)} &\leq C_\lambda(x)\,\lambda^n\,|v_u|_x && (9.1\text{b})\\
\max\{|p^s_x|_x, |p^u_x|_x\} &\leq C_\lambda(x) && (9.1\text{c})
\end{aligned}
$$

for all $x \in X$, $v_s \in \boldsymbol{E}^s_{i(x)}$, $v_u \in \boldsymbol{E}^u_{i(x)}$ and positive n. The last condition states that the splitting of the fibre over the point x, is $C_\lambda(x)$-nondegenerate. It is very important to note that the function C_λ is in general *not* continuous, indeed it might not even be measurable.

9.1.1 Uniform hyperbolicity

A *C-uniformly λ-hyperbolic* invariant set $\Lambda(X, h, i)$ is an invariant set for which there exists a constant C and a function $C_\lambda : X \to [1, \infty)$ such that $C_\lambda(x) \leq C$ for all $x \in X$ and moreover, the set Λ is weakly λ-hyperbolic *with respect to the function C_λ*. Note that we can, and often do, take $C_\lambda(x) = C$. We normally reserve the symbol $\Lambda_{C,\lambda}$ for a uniformly hyperbolic invariant set.

For a uniformly hyperbolic invariant set, the splitting is continuous and can always be continuously extended to a unique splitting defined on the closure of the invariant set [HP70, Shu87]. The continuity of the splitting means that the maps $\boldsymbol{E}^s : \Lambda \to G_\Lambda M$, $\boldsymbol{E}^u : \Lambda \to G_\Lambda M$, and $(\boldsymbol{E}^s \oplus_\Lambda \boldsymbol{E}^u) : \Lambda \to \mathcal{S}_C(T_\Lambda M)$ into the appropriate Grassmannian bundles are all continuous sections.

This continuity of the splitting implies that the maps $x \mapsto p^s_x$, and $x \mapsto p^u_x$ are also continuous. Since M is compact, we can, by considering the closure of an invariant set Λ, see that this implies that the norms of p^s_x and p^u_x are uniformly bounded. This means that condition 9.1c is redundant for *uniformly* hyperbolic invariant sets.

9.1.2 Weak hyperbolicity

Recall that in section 6.3, as part of the definition of C^r_κ continuity of a fibre bundle map, F, of a normed bundle, $\pi : \boldsymbol{E} \to H$, to itself, we defined the concept of a κ-slowly varying function $g : H \to (0, \infty)$. where H is the base space of the bundle. In this definition the function g was slowly varying *relative* to the action of the base map, h, of F. This definition works *because* the base space is an "invariant" set for the base map h.

Given any f-invariant set $\Lambda\,(X, h, i)$, we can make an analogous definition of "slowly varying" relative to the action of the map, f on the set Λ. That is, any function $g : \Lambda \to (0, \infty)$ is *κ-slowly varying* if

$$\frac{1}{\kappa} \leq \frac{g\,(h\,(x))}{g\,(x)} \leq \kappa$$

for all $x \in \Lambda$.

Given a constant $1 \leq h_0$ and a κ-slowly varying function, $C_{\kappa,\lambda}$, of an invariant, $\Lambda\,(x, h, i)$, define $h_n = h_0 \kappa^n$ and $h_{-1} = 0$. Then, for each positive n, we define $X_n = \{x \in X \mid h_{n-1} < C_{\kappa,\lambda}\,(x) \leq h_n\}$.

An invariant set, $\Lambda\,(X, h, i)$, is *κ-λ-hyperbolic with h_0-hyperbolic blocks*, if there exists a κ-slowly varying function, $C_{\kappa,\lambda} : X \to [1, \infty)$ for which the set Λ is weakly λ-hyperbolic with respect to the function $C_{\kappa,\lambda}$, and moreover $X = \bigcup_{n=0}^{\infty} X_n$ where the X_n are defined as above for h_0 and κ. We will normally reserve the symbol $\Lambda_{h_0,\kappa,\lambda}$ to denote a κ-slowly varying hyperbolic invariant set with h_0-hyperbolic blocks. In general when referring to a *hyperbolic set* we will mean an invariant set which is κ-λ-hyperbolic with h_0-hyperbolic blocks for some $0 < \lambda < \frac{1}{\kappa} \leq 1 \leq h_0$. Note that a 1-$\lambda$-hyperbolic invariant set with h_0-hyperbolic blocks is an h_0-uniformly λ-hyperbolic set.

For any κ-λ-hyperbolic invariant set with h_0-hyperbolic blocks, we call each X_n an h_n-hyperbolic block. This system of hyperbolic blocks allows us to discuss "degrees" of hyperbolicity among the points of a κ-λ-hyperbolic set. The points of X_{n+1} satisfy "weaker" hyperbolicity conditions than do the points of X_n. One of the most important consequences of this definition is that, since the function $C_{\kappa,\lambda}$ is κ-slowly varying, then $x \in X_n$ and $h(x) \in X_m$ implies that $n-1 \leq m \leq n+1$.

The system of h_n-hyperbolic blocks for $\Lambda_{h_0,\kappa,\lambda}$ is a partition of X. This partition gives rise to a classification function, $\chi(x) = n$ whenever $x \in X_n$, and associated gradation, $\{P_n\}_0^\infty$, where $P_n = \bigcup_{i=0}^n X_n$. Since this classification function trivially satisfies the slowly varying condition, we can identify the 0-pseudo orbit $\Lambda_{\kappa,\lambda}$ of M and f with a unique classified 0-pseudo orbit $\Lambda_{\kappa,\lambda}$ of M and f, and hence a unique 0-pseudo orbit $\mathbf{\Lambda}_{\kappa,\lambda}$ of $\boldsymbol{M}$ and f. We will often, interchangeably, refer to the elements of either the partition, $\{X_n\}_0^\infty$, *or* the gradation, $\{P_n\}_0^\infty$, as h_0-hyperbolic blocks. Note that since $X_n = \{x \in X \mid h_{n-1} < C_{\kappa,\lambda}(x) \leq h_n\}$, these partitions, gradations, and classification functions are *minimal* with respect to the function $C_{\kappa,\lambda}$

The other important, though slightly less obvious, consequence of the definition of a system of hyperbolic blocks is that the splitting associated with the hyperbolicity of $\Lambda_{h_0,\kappa,\lambda}$ is unique and continuous when restricted to any one hyperbolic block, P_n. More precisely, Pesin [Pes76] showed that, the sections $\boldsymbol{E}^s : \Lambda \to G_\Lambda M$, $\boldsymbol{E}^u : \Lambda \to G_\Lambda M$, and $\boldsymbol{E}^s \oplus_\Lambda \boldsymbol{E}^u : \Lambda \to S_{k_m}(T_\Lambda M)$ are all continuous when restricted to any P_n. We will essentially use Pesin's proof to show essentially the same thing in a slightly more general context in lemma 9.3 below.

9.1.3 Pesin theory

A single orbit is an invariant set and so we can ask whether or not it is κ-λ-hyperbolic. Any orbit which is Lyapunov regular has a unique spectrum of Lyapunov (characteristic) exponents. See [Ose68] or [Pes77] for the relevant definitions. Pesin [Pes76] proved that any regular orbit whose Lyapunov exponents lie outside the interval $[-\ln(\lambda), \ln(\lambda)]$ is κ-λ-hyperbolic with h_0-hyperbolic blocks for all $0 < \lambda < \frac{1}{\kappa} < 1 \leq h_0$.

In general the function $C_{\kappa,\lambda}$ which Pesin constructed (see [Pes76] Theorem

1.1.1) depends crucially on the values of κ and λ chosen. Fix a regular point x and let $\{\chi_i\}_1^{r(x)}$ denote the spectrum of Lyapunov exponents for x with respect to f. Since x is regular, the spectrum of Lyapunov exponents for x with respect to f^{-1} is $\left\{-\chi_{(r(x)-i+1)}\right\}_1^{r(x)}$. Let $\chi^{\pm} = \max\{\pm\chi_i \mid \pm\chi_i < 0\}$, and let $\bar{\lambda} = \exp(\max\{\chi^+, \chi^-\})$. Then, in general, for fixed regular points x, Pesin's function $C_{\kappa,\lambda}$ approaches ∞ as either λ approaches $\bar{\lambda}$ or κ approaches 1.

Let $\hat{\Lambda}_{h_0,\kappa,\lambda}$ denote the set of *all* points in the manifold whose orbit is κ-λ-hyperbolic with h_0-hyperbolic blocks for some κ-slowly varying function $C_{\kappa,\lambda}$. Let

$$\begin{aligned} \hat{\Lambda}_{\lambda} &= \bigcap_{0<\lambda<\frac{1}{\kappa}\leq 1\leq h_0} \hat{\Lambda}_{h_0,\kappa,\lambda}, \quad \text{and} \\ \hat{\Lambda}_{*} &= \bigcup_{0<\lambda<1} \hat{\Lambda}_{\lambda}. \end{aligned}$$

Note that for fixed $\lambda < 1$ we know that $\hat{\Lambda}_{\lambda} \subset \hat{\Lambda}_{h_0,\kappa,\lambda} \subset \hat{\Lambda}_{\bar{h}_0,\bar{\kappa},\lambda}$ for all $\lambda < \frac{1}{\bar{\kappa}} \leq \frac{1}{\kappa} < 1 \leq h_0 \leq \bar{h}_0$. Moreover, it is easy to see that, for $\lambda < \bar{\lambda} < \frac{1}{\kappa} < 1 \leq h_0 \leq \bar{h}_0$ we have that $\hat{\Lambda}_{h_0,\kappa,\lambda} \subset \hat{\Lambda}_{\bar{h}_0,\kappa,\bar{\lambda}}$. This implies that, for $\lambda < \bar{\lambda} < 1$, we know that $\hat{\Lambda}_{\lambda} \subset \hat{\Lambda}_{\bar{\lambda}}$. We will call the set $\hat{\Lambda}_{*}$ the (weakly) hyperbolic Pesin set.

By far the most important consequences of these definitions is that the set of regular points with nonzero Lyapunov exponents is contained in the weakly hyperbolic Pesin set, $\hat{\Lambda}_{*}$. Oseledec [Ose68] proved that the set of regular points has full measure with respect to any f-invariant measure. In particular this means that $\hat{\Lambda}_{*}$ has full measure with respect to any f-invariant measure which has nonzero Lyapunov exponents, and hence there exists at least one $\hat{\Lambda}_{h_0,\kappa,\lambda}$ with nonzero measure for some $\lambda < \frac{1}{\kappa} < 1 \leq h_0$. If the f-invariant measure, μ, is ergodic and its spectrum of Lyapunov exponents lies outside of the interval $\left[-\ln\left(\bar{\lambda}\right), \ln\left(\bar{\lambda}\right)\right]$ for some $\bar{\lambda} < 1$, then Oseledec's theorem implies that the set $\hat{\Lambda}_{h_0,\kappa,\lambda}$ has full μ-measure for *any* choice of $\bar{\lambda} \leq \lambda < \frac{1}{\kappa} < 1 \leq h_0$.

9.2 Generalizations of hyperbolicity to pseudo-orbits

In order to define hyperbolicity for an invariant set Λ, we associated to the invariant set Λ a natural fibre bundle $T_\Lambda M$ which is related to the tangent bundle of M. In order to define hyperbolicity for a pseudo-orbit, we must find a similarly "natural" vector bundle of X. The natural bundle to associate to the index space X of a pseudo-orbit $\mathfrak{A}$ is the pull back via the injection i of the tangent bundle of M, i.e. $i^*T_{i(X)}M$. Since this bundle serves the purpose of a "tangent" bundle, we will call this pull back via i the *tangent bundle of the pseudo-orbit* $\mathfrak{A}$ and denote it by $T\mathfrak{A}$. We note that, in general, any C^r Riemannian metric of M, pulls back to a continuous Riemannian metric on $T\mathfrak{A}$. If i is C^r then the pullback Riemannian metric is also C^r.

For an invariant set Λ, the statement that it is hyperbolic is roughly the statement that

1. its tangent bundle, $T_\Lambda M$, has a splitting into "stable" and "unstable" subbundles,
2. Tf asymptotically contracts sections of the "stable" subbundle, and
3. Tf^{-1} asymptotically contracts sections of the "unstable" subbundle.

Hence, in order to generalize the concept of hyperbolicity to pseudo-orbits, we must identify a suitable linear bundle morphism of the tangent bundle $T\mathfrak{A}$.

The proof of the Stable Manifold theorem, for an either uniformly [HP70, Shu87] or weakly [PS89] hyperbolic invariant set Λ, essentially proceeds by showing that the lift $\mathfrak{f}$ of the diffeomorphism f, via the exp map is close enough to Tf, over appropriately small neighbourhoods of each point in the invariant set, for the contraction of Tf to imply that the lift $\mathfrak{f}$ itself contracts an appropriate space of sections of $T_\Lambda M$. The resulting fixed points of this contraction by the lift $\mathfrak{f}$ correspond, via the exp map, to the local stable and unstable manifolds of points in Λ. The important thing to note is that Tf is the derivative of this lift at the zero section of $T_\Lambda M$.

9.2.1 The importance of the exp map

Any fixed family of charts of the manifold M, together with an associated family of identifications of the fibres of TM with $\mathbb{R}^m$ could be used to lift the diffeomorphism f into the tangent bundle. However, for a Riemannian manifold, *the* most natural way to lift the diffeomorphism f is via the map $\exp_x : T_xM \to M$. We recall some of the more useful facts about the exp map

1. $\exp_x(0) = x, \quad D_0\exp_x = Id_{T_xM}$,

2. $\exp_x$ sends lines in T_xM through the origin to geodesics in M through x,

3. since M is compact there exists an $r > 0$ such that for all $x \in M$, if $B_r(0) \subset T_xM$ we have that $\exp_x : B_r(0) \to M$ is a diffeomorphism,

4. $\exp_x^{-1}(x) = 0, \quad D_x\exp_x^{-1} = Id_{T_xM}$,

5. for all $x \in M$ and $v \in T_xM$ such that $|v| \leq r$ we have $d(\exp_x(v), x) = |v|$.

We call any neighbourhood, U_x, of a point x for which $\exp_x^{-1}$ is a diffeomorphism when restricted to U_x, a normal neighbourhood of x. The most important property of the $\exp_x$ map restricted to such a *normal neighbourhood* of the point x is that the relationship between vectors of the fibre T_xM and geodesics, points, and distances in the manifold, is particularly simple. This will greatly simplify most of the following arguments.

Since the exp map depends on the Riemannian metric used, the exp maps corresponding to two different Riemannian metrics are, in general, different. In order to be able to relate our estimates back to the original Riemannian metric, *we will always use the* exp *map associated to the original metric.*

Let $\mathfrak{r}_0$ be chosen so that for all x the ball $B_{\mathfrak{r}_0}(0) \subset T_xM$ is a normal neighbourhood for $\exp_x$. Then, it will be convenient to fix numbers $\mathfrak{r}_M < 1$ and $\mathfrak{r}_f$ such that, for all $x, y, z \in M$

1. the ball $B_{\mathfrak{r}_M}(x)$ of radius $\mathfrak{r}_M < \mathfrak{r}_0$ contained in M, is a normal neighbourhood of x,

2. if $d(f(y), z) \leq \mathfrak{r}_M$, then the ball $B_{\mathfrak{r}_M}(y)$ in M is mapped, by f, into the ball $B_{\mathfrak{r}_0}(z)$ in M.

3. if $d(f^{-1}(y), z) \leq \mathfrak{r}_M$, then the ball $B_{\mathfrak{r}_M}(y)$ in M is mapped, by f^{-1}, into the ball $B_{\mathfrak{r}_0}(z)$ in M,

4. if $d_{\sup}(f, g) \leq \mathfrak{r}_f$ and $d(g(y), z) \leq \mathfrak{r}_M$ then $g(B_{\mathfrak{r}_M}(y)) \subset B_{\mathfrak{r}_0}(z)$, and

5. if $d_{\sup}(f^{-1}, g^{-1}) \le \mathfrak{r}_f$ and $d(g^{-1}(y), z) \le \mathfrak{r}_M$ then $g^{-1}(B_{\mathfrak{r}_M}(y)) \subset B_{\mathfrak{r}_0}(z)$.

Since f is C^1, exp is C^∞, and M is compact, this can be done. We place these restrictions on $\mathfrak{r}_M$, so that the following definitions of the lift will take place in neighbourhoods of y and z, for which $\exp_y$ and $\exp_z$ are diffeomorphisms. More importantly, we assume, from now on, that *all pseudo-orbits have a perturbation constant less than or equal to* $\mathfrak{r}_M$.

9.2.2 Lifts via the exp map

For a given pseudo-orbit $\mathfrak{A}$ ($\boldsymbol{\mathfrak{A}}$), we can then use the exp map associated to the *original* Riemannian metric to define the lift $\mathfrak{g}$ ($\boldsymbol{\mathfrak{g}}$) of any diffeomorphism g for which $d_{\sup}(g, f) < \mathfrak{r}_f$ in the following way.

If y and z are two points in M with $d(g(y), z) \le \mathfrak{r}_M$, we can define the map

$$\mathfrak{g}_{z,y} : T_yM \to T_zM, \quad \text{by} \quad \mathfrak{g}_{z,y}(v) = \exp_z^{-1} \circ g \circ \exp_y(v),$$

where $v \in T_yM$, and $|v| \le \mathfrak{r}_M$. The image of the vector v in the tangent space at y is the result of, first mapping the vector down onto the manifold via the map $\exp_y$, then mapping the resulting point via the diffeomorphism g, and finally mapping the resulting point in the manifold into the corresponding vector in the tangent space of the point z. See figure 9.1. We require that $d_{\sup}(g, f) \le \mathfrak{r}_f$, $d(g(y), z) \le \mathfrak{r}_M$ and $|v| \le \mathfrak{r}_M$, in order to ensure that the point $g\left(\exp_y(v)\right)$ is contained in a normal neighbourhood of z.

It will be convenient to note that the inverse of the lift $\mathfrak{g}$, i.e. $\mathfrak{g}_{z,y}^{-1} : T_zM \to T_yM$, is defined by $\mathfrak{g}_{z,y}^{-1}(v) = \exp_y^{-1} \circ g^{-1} \circ \exp_z(v)$, for $v \in T_zM$, and $|v| \le \mathfrak{r}_M$.

The action of the lift $\mathfrak{g}$ in the fibre of $T\mathfrak{A}$ over the point $x \in X$, is then defined as

$$\begin{aligned} \mathfrak{g}_x(v) &= \mathfrak{g}_{ih(x), i(x)}(v), \quad \text{that is} \\ \mathfrak{g}_x(v) &= \exp_{ih(x)}^{-1} \circ g \circ \exp_{i(x)}(v), \end{aligned}$$

where, again, $v \in T_yM$, and $|v| \le \mathfrak{r}_M$. The lift $\mathfrak{g}$ is then defined so as to make the diagram

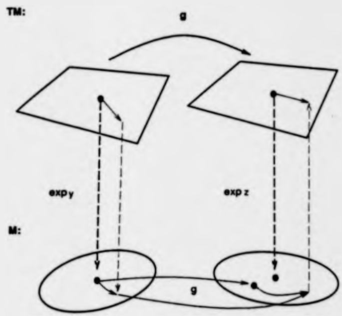

Figure 9.1: Lifting a diffeomorphism from the manifold to its tangent bundle.

$$
\begin{tikzcd}
\Delta_{\tau_M} T\mathfrak{A} \arrow[r, "\mathfrak{g}"] \arrow[d, dash] & T\mathfrak{A} \arrow[d, dash] \\
X \arrow[r, "h"] & X
\end{tikzcd}
$$

commute.

Recall that, if the perturbation constant of the pseudo-orbit $\mathfrak{A}$ is zero, i.e. $\mathfrak{p}(\mathfrak{A}) = 0$, then $\mathfrak{A}$ is really an invariant set. In this case our lift $\mathfrak{g}$ corresponds to the usual concept of the lift of the diffeomorphism, g, over an invariant set.

9.2.3 The derivative of the lift $\mathfrak{f}$

For a pseudo-orbit $\mathfrak{A}$ ($\boldsymbol{\mathfrak{A}}$) of the diffeomorphism f of M, the most important lift to consider is the lift $\mathfrak{f}$ ($\boldsymbol{\mathfrak{f}}$) of the diffeomorphism f ($\boldsymbol{f}$) itself. Indeed we define the hyperbolicity of a pseudo-orbit in terms of the hyperbolicity of the derivative of $\mathfrak{f}$ at the zero section of $T\mathfrak{A}$. Let $F = T_0\mathfrak{f}$ ($\boldsymbol{F} = T_0\boldsymbol{\mathfrak{f}}$). Then the bundle map $F : T\mathfrak{A} \to T\mathfrak{A}$ ($\boldsymbol{F} : T\boldsymbol{\mathfrak{A}} \to T\boldsymbol{\mathfrak{A}}$) is a linear tangent morphism of the tangent bundle, $T\mathfrak{A}$ ($T\boldsymbol{\mathfrak{A}}$), of pseudo-orbit $\mathfrak{A}$ ($\boldsymbol{\mathfrak{A}}$).

In order to give an explicit formula for F, we start by calculating the derivatives of $\mathfrak{f}_{z,y}$ and $\mathfrak{f}_{z,y}^{-1}$. For v in T_zM, the derivative of $\exp_z$ at v, i.e. $D_v\exp_z$, maps T_vT_zM to $T_{\exp_z(v)}M$. Since T_zM is itself a linear space there is a canonical isomorphism which identifies T_vT_zM and T_zM. As above, if y and z are two points in M with $d(f(y), z) \leq \tau_M$, we can define the map

$$F_{z,y} : T_yM \to T_zM, \quad \text{by} \quad F_{z,y} = D_{f(y)}\exp_z^{-1} D_y f.$$

The inverse $F_{z,y}^{-1} : T_zM \to T_yM$, is defined by $F_{z,y}^{-1} = D_{f(y)}f^{-1} D_{\exp_z^{-1}(f(y))}\exp_z$. In terms of $\mathfrak{f}_{z,y}$ and $\mathfrak{f}_{z,y}^{-1}$ we have

$$F_{z,y} = D_0\mathfrak{f}_{z,y}, \qquad F_{z,y}^{-1} = D_{\exp_z^{-1}(f(y))}\mathfrak{f}_{z,y}^{-1}.$$

The action of F in the fibre of $T\mathfrak{A}$ over the point $x \in X$, is then defined as

$$
\begin{aligned}
F_x(v) &= F_{ih(x),i(x)}(v), \quad \text{that is} \\
F_x(v) &= D_{f(i(x))}\exp_{ih(x)}^{-1} D_{i(x)} f(v)
\end{aligned}
$$

where, $v \in T_{i(x)}M$. The inverse action is given by

$$\begin{aligned} F_x^{-1}(v) &= F^{-1}_{ih(x),i(x)}(v), \quad \text{or} \\ F_x^{-1}(v) &= D_{fi(x)}f^{-1}D_{\exp^{-1}_{ih(x)}(fi(x))}\exp_{ih(x)}(v) \end{aligned}$$

where, $v \in T_{ih(x)}M$.

Again, the morphisms F and F^{-1} are then defined so that the diagrams

$$\begin{tikzcd} T\mathfrak{A} \arrow[d] & \Delta_{\tau_M}T\mathfrak{A} \arrow[l, "F^{-1}"'] \arrow[r, "F"] \arrow[d, dash] & T\mathfrak{A} \arrow[d, dash] \\ X & X \arrow[l, "h^{-1}"'] \arrow[r, "h"] & X \end{tikzcd}$$

commute.

If $\mathfrak{A}$ is an τ_M-pseudo-orbit then we have

$$F_x = D_0\mathfrak{f}_x, \qquad F_x^{-1} = D_{\exp_x^{-1}(f(y))}\mathfrak{f}_x^{-1}.$$

Note that the apparent asymmetry between the definitions of F and its inverse reflects the inherent asymmetry implied in the definition of a pseudo-orbit. As noted in the previous chapter, this inherent asymmetry is best seen in the fact that while $\mathfrak{A}$ might be an α-pseudo orbit of M and f ((α_n)-pseudo orbit of M and f), the corresponding inverse pseudo-orbit, $\mathfrak{A}^{-1}$, of M and f^{-1} need not be an α-pseudo orbit ((α_n)-pseudo orbit).

Since M is compact, there exists a number $K > 0$, the Lipschitz constant of f^{-1}, such that the inverse pseudo-orbit, $\mathfrak{A}^{-1}$, of M and f^{-1} which corresponds to any α-pseudo orbit ((α_n)-pseudo orbit) $\mathfrak{A}$ of M and f is a $K\alpha$-pseudo orbit ($(K\alpha_n)$-pseudo orbit). Note that the lift $\mathfrak{f}^{-1}$ is the lift of f^{-1} associated to the inverse pseudo-orbit $\mathfrak{A}^{-1}$. However, it is important to note that F^{-1} does *not* correspond to the derivative of $\mathfrak{f}^{-1}$ at the zero section of $T\mathfrak{A}^{-1} = T\mathfrak{A}$.

9.2.4 Lifts $\mathfrak{g}$ Lipschitz close to F

Pseudo-orbits and in particular the (pseudo) hyperbolic pseudo-orbits to be defined in the next two sections are particularly well suited to applications of the perturbed Stable manifold theorem proven in Part II. One of the most important assumptions of that theorem is that the bundle map and its inverse are Lipschitz close to a linear hyperbolic bundle map relative to a κ-slowly varying disc bundle. In all of our applications the linear hyperbolic bundle map will be the tangent morphism, F, of some (pseudo) hyperbolic pseudo-orbit $\mathfrak{A}$, and moreover, in all of our applications the bundle map will be the lift of some diffeomorphism $\mathfrak{r}_f$ close to f.

We are then interested in showing that the lift of a diffeomorphism, g, which is $\mathfrak{r}_f$ close to f is Lipschitz over some slowly varying disc bundle. We will show, in this chapter, that this can be done in the next two lemmas. These two cases are really merely preludes to the more interesting slowly varying versions of these same lemmas to be proven for the classifying manifold M. Those slowly varying lemmas will be dealt with in the next chapter.

Lemma 9.1 *Consider a C^1 diffeomorphism f, and an $\varepsilon > 0$. Then there exists a constant $\mathfrak{N}$ and a C^1 neighbourhood $V = V(\varepsilon)$ of f such that if $g \in V$ and $\mathfrak{A}$ is any (classified) $\mathfrak{r}_M$-pseudo orbit of M and f, then we have*

$$\begin{aligned} Lip\left([F - \mathfrak{g}]\big|_{\triangle_{\mathfrak{N}} T\mathfrak{A}}\right) &\leq \varepsilon, \\ Lip\left(\left[F^{-1} - \mathfrak{g}^{-1}\right]\Big|_{\triangle_{\mathfrak{N}} T\mathfrak{A}}\right) &\leq \varepsilon. \end{aligned}$$

Moreover, since $\exp$ *is C^∞ and g is C^1, for all $x \in X$ the lift, $\mathfrak{g}_x$, is C^1. If $\mathfrak{A}$ is C^0_* then so is the lift* $\mathfrak{g}$.

Proof: Since the perturbation constant of $\mathfrak{A}$ is at most $\mathfrak{r}_M$, for any $x \in X$, the map $\mathfrak{g}_x$ is defined on the ball $B_{\mathfrak{r}_M}(0)$ with respect to the original metric. For $v \in B_{\mathfrak{r}_M}(0)$ we have

$$\begin{aligned} & D_v(F_x - \mathfrak{g}_x) \\ &\quad = D_v\left[D_{f\circ i(x)}\left(\exp^{-1}_{h\circ i(x)}\right) D_{i(x)} f - \exp^{-1}_{h\circ i(x)} g \exp_{i(x)}\right] \\ &\quad = D_{f\circ i(x)}\left(\exp^{-1}_{h\circ i(x)}\right) D_{i(x)} f \\ &\qquad - D_{g(\exp_{i(x)}(v))}\left(\exp^{-1}_{h\circ i(x)}\right) D_{\exp_{i(x)}(v)}(g) D_v\left(\exp_{i(x)}\right). \end{aligned}$$

This is defined and equal to zero for $g = f$ and $v = 0$. Since M is compact and this derivative is continuous in both v and g, we know that there exists a C^1

neighbourhood, V_+, of f and a constant, $\mathfrak{R}_+$, for which if $g \in V_+$ we have

$$Lip\left(\left[F - \mathfrak{g}\right]\Big|_{\Delta_{\mathfrak{R}_+} T\mathfrak{A}}\right) \leq \varepsilon,$$

where $\mathfrak{g}$ is the lift of g relative to the pseudo-orbit $\mathfrak{A}$. Similar arguments show that there exists a C^1 neighbourhood, V_-, of f and a constant, $\mathfrak{R}_-$, for which if $g \in V_-$ we have

$$Lip\left(\left[F^{-1} - \mathfrak{g}^{-1}\right]\Big|_{\Delta_{\mathfrak{R}_-} T\mathfrak{A}}\right) \leq \varepsilon,$$

where, again, $\mathfrak{g}$ is the lift of g relative to the pseudo-orbit $\mathfrak{A}$.

We are done if we let $V = V_+ \bigcap V_-$ and $\mathfrak{R} = \min\{\mathfrak{R}_+, \mathfrak{R}_-\}$. ∎

Lemma 9.2 *Consider a $C^{1+\gamma}$ diffeomorphism f, and an $\varepsilon > 0$. Then there exists a positive constant C_f such that for any constant $0 < \mathfrak{R} < \mathfrak{r}_M \leq 1$ we have*

$$\begin{aligned} Lip\left(\left[F - \mathfrak{f}\right]\Big|_{\Delta_{\mathfrak{R}} T\mathfrak{A}}\right) &\leq C_f \mathfrak{R}^{\gamma}, \\ Lip\left(\left[F^{-1} - \mathfrak{f}^{-1}\right]\Big|_{\Delta_{\mathfrak{R}} T\mathfrak{A}}\right) &\leq C_f \mathfrak{R}^{\gamma}. \end{aligned}$$

In particular we can choose $\mathfrak{R}$ so small that $C\mathfrak{R}^{\gamma} < \varepsilon$. Moreover, since exp *is C^{∞} and f is $C^{1+\gamma}$, for all $x \in X$ the lift, $\mathfrak{f}_x$, is $C^{1+\gamma}$. If $\mathfrak{A}$ is C^0_* then so is the lift $\mathfrak{f}$.*

Proof: Recall that $\mathfrak{f}_{z,y} : T_y M \to T_z M$ is defined as $\mathfrak{f}_{z,y} = \exp_z^{-1} \circ f \circ \exp_y$ for all $d(z, f(y)) \leq \mathfrak{r}_M$. By applying the fundamental theorem of calculus to the function $\mathfrak{f}_{z,y}$ from the Banach space $T_y M$ to the Banach space $T_z M$ for $u, w \in B_{\mathfrak{r}_M}(v) \subset T_y M$ along the path $t \mapsto (1-t)w + tu$ we have

$$\mathfrak{f}_{z,x}(u) - \mathfrak{f}_{z,x}(w) = \left[\int_0^1 D_{(1-t)w+tu}\mathfrak{f}_{z,x}\, dt\right](u - w),$$

and so

$$\mathfrak{f}_{z,x}(u) - \mathfrak{f}_{z,x}(w) - D_v \mathfrak{f}_{z,y} = \left[\int_0^1 D_{(1-t)w+tu}\mathfrak{f}_{z,x} - D_v \mathfrak{f}_{z,y}\, dt\right](u - w).$$

Since exp, and $\exp^{-1}$ are C^{∞} and f and f^{-1} are $C^{1+\gamma}$, we know that both $\mathfrak{f}_{z,y}$ and $\mathfrak{f}_{z,y}^{-1}$ are $C^{1+\gamma}$ for all $d(z, f(y)) \leq \mathfrak{r}_M$. Since M is compact there exists a positive constant, C, for which $\mathfrak{f}_{z,y}$ and $\mathfrak{f}_{z,y}^{-1}$ are uniformly γ-Hölder continuous over the set $\{(y,z) \in \mathcal{M} \times \mathcal{M} \mid d(f(y), z) \le \mathfrak{r}_{\mathcal{M}}\}$. By taking norms, and noting that for all $\mathfrak{R} \le \mathfrak{r}_M \le 1$ and $u, w \in B_{\mathfrak{R}}(v)$ we have $|((1-t)w + tu) - v| \le \mathfrak{R}$, we have

$$\left\| D_{((1-t)w+tu)}\mathfrak{f}_{z,y} - D_v\mathfrak{f}_{z,y} \right\| \le C\,|((1-t)w + tu) - v|^{\gamma} \le C\mathfrak{R}^{\gamma}$$

and hence

$$\begin{aligned} &|\mathfrak{f}_{z,y}(u) - \mathfrak{f}_{z,y}(w) - D_0\mathfrak{f}_{z,y}(u-w)| \\ &\quad \le \left[\int_0^1 \left| D_{((1-t)w+tu)}\mathfrak{f}_{z,y} - D_0\mathfrak{f}_{z,y} \right| dt \right] |u-w| \\ &\quad \le C\mathfrak{R}^{\gamma}\,|u-w| \end{aligned}$$

Similar arguments applied to $\mathfrak{f}_{z,y}^{-1} = \exp_z^{-1} \circ f^{-1} \circ \exp_z$ shows that

$$\left| \mathfrak{f}_{z,y}^{-1}(u) - \mathfrak{f}_{z,y}^{-1}(w) - D_{\exp_z^{-1}(f(y))}\mathfrak{f}_{z,y}^{-1}(u-w) \right| \le C\mathfrak{R}^{\gamma}\,|u-w|$$

We are done after noting that $F_x = F_{ih(x),i(x)} = D_0\mathfrak{f}_{ih(x),i(x)}$, $F_x^{-1} = F_{ih(x),i(x)}^{-1} = D_{\exp_{ih(x)}^{-1}(fi(x))}\mathfrak{f}_{ih(x),i(x)}^{-1}$, and that $\mathfrak{p}(\mathfrak{A}) = d_{\sup}(ih(x), fi(x)) \le \mathfrak{r}_M$. ■

9.2.5 Hyperbolic pseudo-orbits

Given a pseudo-orbit $\mathfrak{A}$ for the manifold M and the diffeomorphism f, the three most important objects which can be associated to the pseudo-orbit $\mathfrak{A}$, are the tangent bundle $T\mathfrak{A}$, the lift, $\mathfrak{f}$, of f into $T\mathfrak{A}$, and the derivative, F, of the lift at the zero section of $T\mathfrak{A}$. Both the Stable Manifold theorem and the Shadowing Lemma are consequences of the "almost hyperbolic" nature of the lift $\mathfrak{f}$ as it acts on sections of "stable" and "unstable" subbundles of the tangent bundle $T\mathfrak{A}$. The lift $\mathfrak{f}$ is "almost hyperbolic" *because* of its proximity to the *hyperbolic* derivative F. Hence the following definitions of hyperbolicity for pseudo-orbits, are no more nor less than what is required to extend these two theorems to pseudo-orbits.

Recall that in order to define a (factored) pseudo-orbit of $\boldsymbol{M}$ and f, we defined pseudo-orbits of M and f, unfactored pseudo-orbits of $\boldsymbol{M}$ and f and then, finally, we defined when a pseudo-orbit of M and f was a factor of a pseudo-orbit of $\boldsymbol{M}$ and f. It was only after all of these definitions that we were able to define a (factored) pseudo-orbit of $\boldsymbol{M}$ and f. We repeat exactly the same steps in order to define the hyperbolicity of a (factored) pseudo-orbit.

Recall that given h_0 and κ we have defined $h_n = h_0\kappa^n$. By essentially repeating the definition of hyperbolicity for invariant sets given above, we define a C^0_* classified pseudo-orbit, $\mathfrak{A}(X, h, i, \chi)$, of M and f, to be (C^0_*) *κ-λ-hyperbolic with h_0-hyperbolic blocks* for $0 < \lambda < \frac{1}{\kappa} \leq 1$ and $1 \leq h_0$, if there exists an F-invariant splitting $T\mathfrak{A} = \boldsymbol{E}^s_X \oplus_X \boldsymbol{E}^u_X$, for which

$$
\begin{aligned}
\left|F^m_x(v_s)\right| &\leq h_n\lambda^m\left|v_s\right| && (9.2a)\\
\left|F^{-m}_x(v_u)\right| &\leq h_n\lambda^m\left|v_u\right| && (9.2b)\\
\max\left\{\|p^s_x\|, \|p^u_x\|\right\} &\leq h_n && (9.2c)
\end{aligned}
$$

for all $m, n \geq 0$, $x \in X_n$, $v_s \in \boldsymbol{E}^s_x$ and $v_u \in \boldsymbol{E}^u_x$. For any n, we call the X_n (or P_n) an *h_n-hyperbolic block of $\mathfrak{A}$*. Furthermore, we assume that for each n, P_n is a closed subset of X, and for each P_n the functions $\boldsymbol{E}^s$, $\boldsymbol{E}^u : P_n \to G_{i(P_n)}M$, $\boldsymbol{E}^s \oplus_{P_n} \boldsymbol{E}^u : P_n \to S_{h_n}(T_{P_n}M)$, and $F : T_{P_n}\mathfrak{A} \to T_{P_n}\mathfrak{A}$ are each C^0.

An C^0_* unfactored pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, is (C^0_*) *κ-λ-hyperbolic with h_0-hyperbolic blocks* for $0 < \lambda < \frac{1}{\kappa} \leq 1$ and $1 \leq h_0$, if there exists an $\boldsymbol{F}$-invariant splitting $T\boldsymbol{\mathfrak{A}} = \boldsymbol{E}^s_{\boldsymbol{X}} \oplus_{\boldsymbol{X}} \boldsymbol{E}^u_{\boldsymbol{X}}$, for which

$$
\begin{aligned}
\left|\boldsymbol{F}^m_x(v_s)\right| &\leq h_n\lambda^m\left|v_s\right| && (9.3a)\\
\left|\boldsymbol{F}^{-m}_x(v_u)\right| &\leq h_n\lambda^m\left|v_u\right| && (9.3b)\\
\max\left\{\|p^s_x\|, \|p^u_x\|\right\} &\leq h_n && (9.3c)
\end{aligned}
$$

for all $m, n \geq 0$, $x \in \boldsymbol{X}_n$, $v_s \in \boldsymbol{E}^s_x$ and $v_u \in \boldsymbol{E}^u_x$. Furthermore we assume that for each n, the functions $\boldsymbol{E}^s$, $\boldsymbol{E}^u : P_n \to G_{i(P_n)}M$, $\boldsymbol{E}^s \oplus_{P_n} \boldsymbol{E}^u : P_n \to S_{h_n}(T_{P_n}M)$, and $\boldsymbol{F} : T_{P_n}\boldsymbol{\mathfrak{A}} \to T_{P_{n+1}}\boldsymbol{\mathfrak{A}}$ are each C^0.

Figure 9.2 shows a geometric picture of what a hyperbolic pseudo-orbit looks like.

A C^0_* factored pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, is (C^0_*) *κ-λ-hyperbolic with h_0-hyperbolic blocks* for $0 < \lambda < \frac{1}{\kappa} \leq 1$ and $1 \leq h_0$, if

1. the factor pseudo-orbit, $\mathfrak{A}(X, h, i, \chi)$, is a C^0_* κ-λ-hyperbolic with h_0-hyperbolic blocks,

2. the unfactored pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, is a C^0_* κ-λ-hyperbolic with h_0-hyperbolic blocks,

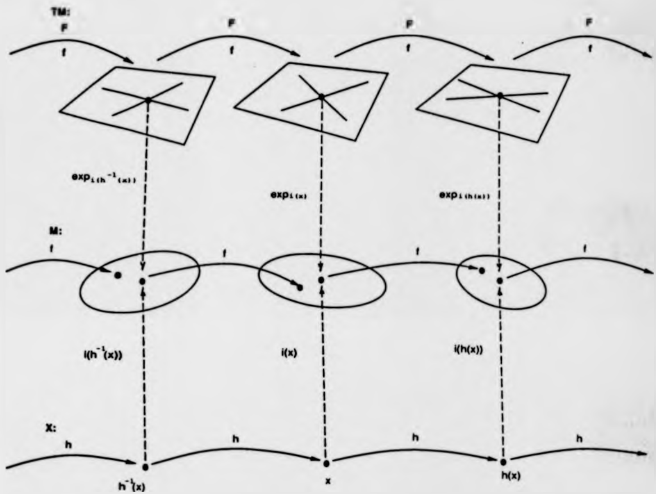

Figure 9.2: A "picture" of an (unfactored) Hyperbolic Pseudo-orbit of M and f.

3. the following diagram commutes

$$\begin{tikzcd}
\boldsymbol{\mathfrak{A}} \arrow[d, "k"'] & T\boldsymbol{\mathfrak{A}} \arrow[l, "\tau_{\boldsymbol{\mathfrak{A}}}"'] \arrow[r, "\boldsymbol{F}"] \arrow[d, "i^*T\pi_M"'] & T\boldsymbol{\mathfrak{A}} \arrow[r, "\boldsymbol{i}"] \arrow[d, "i^*T\pi_M"'] & T\boldsymbol{M} \arrow[d, "T\pi_M"'] \\
\mathfrak{A} & T\mathfrak{A} \arrow[l, "\tau_{\mathfrak{A}}"'] \arrow[r, "F"] & T\mathfrak{A} \arrow[r, "i"] & TM
\end{tikzcd}$$

where we recall that

- $T\mathfrak{A} = i^*T_{i(X)}M$,
- $T\boldsymbol{\mathfrak{A}} = \boldsymbol{i}^*T_{\boldsymbol{i}(\boldsymbol{X})}\boldsymbol{M}$,
- the maps $i : T\mathfrak{A} \to TM$ and $\boldsymbol{i} : T\boldsymbol{\mathfrak{A}} \to T\boldsymbol{M}$ are the extended maps as discussed under the pull back bundle constructions, (see section 3.2 of chapter 3),
- if for $x \in M$ and $(n, x) \in \boldsymbol{M}$ we denote $v \in T_{(n,x)}\boldsymbol{M}$ by (n, x, v) then $T\pi_M$ is defined by $T\pi_M(n, x, v) = (x, v) \in T_xM$,
- if for $x \in X$ and $(n, x) \in \boldsymbol{X}$ we denote $v \in T_{(n,x)}\boldsymbol{\mathfrak{A}}$ by (n, x, v) then $i^*T\pi_M$ is defined by $i^*T\pi_M(n, x, v) = (x, v) \in T_x\mathfrak{A}$.

Finally, a factored hyperbolic pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{k}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, is *aligned* if

$$\begin{aligned}
i^*T\pi_M(\boldsymbol{E}^s_{\boldsymbol{X}}) &= E^s_X, \quad \text{and} \\
i^*T\pi_M(\boldsymbol{E}^u_{\boldsymbol{X}}) &= E^u_X.
\end{aligned}$$

The next lemma states that all factored hyperbolic pseudo-orbits of $\boldsymbol{M}$ and f are aligned. However, as we shall see in the next section, this is *not* true for pseudo-hyperbolic pseudo-orbits.

Lemma 9.3 *If $\boldsymbol{\mathfrak{A}}$ is a κ-λ-hyperbolic factored pseudo-orbit of $\boldsymbol{M}$ and f for $0 < \lambda < \frac{1}{\kappa} \leq 1$ and $\boldsymbol{\mathfrak{A}}$ is C^0_* as a pseudo-orbit, then $\boldsymbol{\mathfrak{A}}$ is an aligned C^0_* hyperbolic pseudo-orbit of $\boldsymbol{M}$ and f.*

Proof: This is essentially the proof used by Pesin [Pes76] to show the continuity of the Df-invariant splitting of the Pesin set. Note that the lemma does *not* assume that the given splitting is continuous (C^0_*). The second part of this lemma will show that the splitting must in fact be continuous.

We begin by showing that for a single $\boldsymbol{h}$-orbit $\{\boldsymbol{h}^n(x)\}_{-\infty}^{\infty}$ of $\mathfrak{A}$ the $\boldsymbol{F}$-invariant splitting along this orbit must be unique. Assume that for each n there are two distinct splittings $T_{\boldsymbol{h}^n(x)}\mathfrak{A} = \boldsymbol{E}^s_{\boldsymbol{h}^n(x)} \oplus \boldsymbol{E}^u_{\boldsymbol{h}^n(x)} = \tilde{\boldsymbol{E}}^s_{\boldsymbol{h}^n(x)} \oplus \tilde{\boldsymbol{E}}^u_{\boldsymbol{h}^n(x)}$ which are both $\boldsymbol{F}$-invariant and which both satisfy the same κ-λ-hyperbolicity conditions along the given pseudo-orbit.

Assume that $x \in \boldsymbol{X}_n$, and let $h_n = h_0\kappa^n$. By the slowly varying condition on $\boldsymbol{h}$ we know that $h^m(x) \in \boldsymbol{P}_{n+|m|}$ for all $m \in \mathbb{Z}$.

Without loss of generality we can assume that since the splittings are distinct there exists a vector v for which (for example) $|v| > 0$, $v \in \boldsymbol{E}^s_x$, and $v \notin \tilde{\boldsymbol{E}}^s_x$. Since $v \notin \tilde{\boldsymbol{E}}^s_x$ we know $v = \tilde{v}_s + \tilde{v}_u$ for $\tilde{v}_s \in \tilde{\boldsymbol{E}}^s_x$, $0 < \tilde{v}_u \in \tilde{\boldsymbol{E}}^u_x$. This implies that there exists a finite positive constant $K = \frac{|v|}{|\tilde{v}_u|}$ for which $|v| \leq K\,|\tilde{v}_u|$. Since the splitting, $\tilde{\boldsymbol{E}}^s_x \oplus \tilde{\boldsymbol{E}}^u_x$, is h_n-non-degenerate, we also know that $|\tilde{v}_u| \leq h_n\,|v|$. Since $0 < \tilde{v}_u \in \tilde{\boldsymbol{E}}^u_x$ we know that, for all $0 \leq m$, we have $|\tilde{v}_u|_x \leq h_n\kappa^m\lambda^m\,|\boldsymbol{F}^m_x(\tilde{v}_u)|_{\boldsymbol{h}^m(x)}$. Similarly, Since $v \in \boldsymbol{E}^s_x$ we know that, for all $0 \leq m$, we have $|\boldsymbol{F}^m_x(v)|_{\boldsymbol{h}^m(x)} \leq h_n\kappa^m\lambda^m\,|v|_x$. All together these four inequalities imply that

$$|v|_x \leq K h_n^3 \kappa^{2m} \lambda^{2m}\,|v|_x$$

for all $0 \leq m$. Since $\lambda\kappa < 1$, this implies that $|v|_x = 0$, which in turn implies that the pair of $\boldsymbol{F}$-invariant splittings along the orbit $\{\boldsymbol{h}^n(x)\}_{-\infty}^{\infty}$ could not be distinct.

Now since $\boldsymbol{F} \circ i^*T\pi_M = i^*T\pi_M \circ \boldsymbol{F}$ and since the factor map k from $\mathfrak{A}$ to $\mathfrak{A}$ is monotonic, if the hyperbolic pseudo-orbit, $\mathfrak{A}$, is *not* aligned, then we can use $i^*T\pi_M$ to pull back a κ-λ-hyperbolic F-invariant splitting of $\mathfrak{A}$ to a κ-λ-hyperbolic $\boldsymbol{F}$-invariant splitting of $\mathfrak{A}$ which is *distinct* from the original κ-λ-hyperbolic $\boldsymbol{F}$-invariant splitting of $\mathfrak{A}$, along some orbit of $\boldsymbol{h}$. Since we have just shown that this is impossible we know that the hyperbolic pseudo-orbit, $\mathfrak{A}$, is aligned.

We must now show that the given splitting is in fact C^0_* continuous. This continuity argument is based on that given by Pesin [Pes76][Theorem 1.3.1]. Since we know that $\mathfrak{A}$ is both factored *and* aligned, we can work with the factored pseudo-orbit, $\mathfrak{A}$, of M and f. Fix $n \geq 0$, consider a point x in M and a sequence of points $x_j \in P_n$ for which $i(x_j) \to x$ as j tends to ∞. Since the index map, i, is injective we can abuse our notation by identifying the point $i(x_j) \in M$ with $x_j \in X$. Since M is compact the convergent sequence, x_j is a Cauchy sequence. Since i is continuous and h is a homeomorphism of X, we know that both $i(h(x_j))$ and $i(h^{-1}(x_j))$ are both Cauchy sequences as well. Let x_+ and x_- denote the convergent point, in M, of the sequences $i(h(x_j))$ and $i(h^{-1}(x_j))$ respectively. Fix $m > 0$. Then similar arguments show that $i(h^m(x_j))$ $(i(h^{-m}(x_j)))$ is a Cauchy sequence which converges to the point $x_{(+,m)} \in M$ $\left(x_{(-,m)} \in M\right)$.

Again, since M is compact, so is the Grassmannian bundle, GM, of M. This means that any sequence in GM has a convergent subsequence. That is, the sequence of subspaces, $E^s_{x_j}$ and $E^u_{x_j}$, have a convergent subsequence. Choose one such convergent subsequence and relabel it so that $\dim\left(E^s_{x_j}\right) = k$ and $\dim\left(E^u_{x_j}\right) = \dim(M) - k$, and $E^s_{x_j} \to E^s_x \subset T_xM$ and $E^u_{x_j} \to E^u_x \subset T_xM$ as j tends to ∞.

Since the dimensions of the $E^s_{x_j}$ are constant and moreover since $E^s_{x_j} \to E^s_x$ we know that $p^s_{x_j} \to p^s_x$, and similarly, $p^u_{x_j} \to p^u_x$. This means that, from condition 9.3c, $\max\{\|p^s_x\|, \|p^u_x\|\} \leq h_n$. In particular this means that $T_xM = E^s_x \oplus E^u_x$.

Since both $F_{z,y}$ and $F^{-1}_{z,y}$ are continuous in z and y, we know that

$$
\begin{aligned}
F_{x_j} = F_{ih(x_j),i(x_j)} &\to F_{x_+,x} = F_x \text{ and} \\
F^{-1}_{x_j} = F^{-1}_{ih^{-1}(x_j),i(x_j)} &\to F^{-1}_{x_-,x} = F^{-1}_x.
\end{aligned}
$$

Now fix $m > 0$. Similar arguments show that

$$
\begin{aligned}
F^m_{x_j} &\to F^m_x \text{ and} \\
F^{-m}_{x_j} &\to F^{-m}_x.
\end{aligned}
$$

Consider $v_s \in E^s_x$, $v_{s,j} \in E^s_{x_j}$ for which $v_{s,j} \to v_s$. Since $F^m_{x_j} \to F^m_x$ we know that $F^m_{x_j}(v_{s,j}) \to F^m_x(v_s)$. This, combined with condition 9.3a applied to the x_j, implies that

$$
|F^m_x(v_s)| \leq h_n \lambda^m |v_s|.
$$

Similar arguments for $v_u \in \boldsymbol{E}^u_{\tilde{x}}$, $v_{u,j} \in \boldsymbol{E}^u_{\tilde{x}_j}$ for which $v_{u,j} \to v_u$ show that

$$\left|\boldsymbol{F}_{\tilde{x}}^{-m}(v_u)\right| \leq h_u \lambda^m |v_u| .$$

∎

In particular, this lemma shows that the *maximally shifted closure* of any κ-λ-hyperbolic invariant set is itself an aligned C^0_{-} κ-λ-hyperbolic invariant set.

If $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \chi, \boldsymbol{f})$ and $\boldsymbol{\mathfrak{B}}\left(\mathfrak{B}, \tilde{k}, \tilde{\boldsymbol{X}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{i}}, \tilde{\chi}, \tilde{\boldsymbol{f}}\right)$ are both hyperbolic pseudo-orbits of $\boldsymbol{M}$ and f, and if $\boldsymbol{\mathfrak{B}}$ is a sub-pseudo-orbit of $\boldsymbol{\mathfrak{A}}$, then the arguments used in lemma 9.3 can be used to show that

$$T\mathfrak{B} = T_{\tilde{X}}\mathfrak{A}, \quad E^s_{\mathfrak{B}} = E^s_{\mathfrak{A}|_{\tilde{X}}}, \quad E^u_{\mathfrak{B}} = E^u_{\mathfrak{A}|_{\tilde{X}}},$$
$$T\boldsymbol{\mathfrak{B}} = T_{\tilde{\boldsymbol{X}}}\boldsymbol{\mathfrak{A}}, \quad \boldsymbol{E}^s_{\boldsymbol{\mathfrak{B}}} = \boldsymbol{E}^s_{\boldsymbol{\mathfrak{A}}|_{\tilde{\boldsymbol{X}}}}, \quad \boldsymbol{E}^u_{\boldsymbol{\mathfrak{B}}} = \boldsymbol{E}^u_{\boldsymbol{\mathfrak{A}}|_{\tilde{\boldsymbol{X}}}}.$$

More generally, if $\boldsymbol{\mathfrak{B}}$ is embedded in $\boldsymbol{\mathfrak{A}}$ as pseudo-orbits of $\boldsymbol{M}$ and f via the injective functions $k_e : \tilde{X} \to X$, and $\boldsymbol{k}_e : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$, then we can define an injective function $T\boldsymbol{k}_e : T\boldsymbol{\mathfrak{B}} \to T\boldsymbol{\mathfrak{A}}$ as follows. For each $(n, x) \in \tilde{\boldsymbol{X}}$ let $(m, y) = \boldsymbol{k}_e(n, x)$ and note that $y = k_e(x)$. Define

$$T_{(n,x)}\boldsymbol{k}_e = \boldsymbol{i}^{-}T_{\tilde{\boldsymbol{i}}(n,x)}Id_{(m,n)}.$$

For $(n, x, v) \in T\boldsymbol{\mathfrak{B}}$ this definition means that

$$T_{(n,x)}\boldsymbol{k}_e(n, x, v) = (m, y, v) .$$

Note that this definition of $T\boldsymbol{k}_e$ makes the following pair of diagrams commute

$$\begin{tikzcd}
T\boldsymbol{\mathfrak{A}} \arrow[r, "\boldsymbol{F}"] & T\boldsymbol{\mathfrak{A}} \arrow[dr, "\boldsymbol{i}"] & \\
 & & T\boldsymbol{M} \\
T\boldsymbol{\mathfrak{B}} \arrow[uu, "T\boldsymbol{k}_e"] \arrow[r, "\tilde{\boldsymbol{F}}"] & T\boldsymbol{\mathfrak{B}} \arrow[uu, "T\boldsymbol{k}_e"'] \arrow[ur, "\tilde{\boldsymbol{i}}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
T\boldsymbol{\mathfrak{A}} \arrow[d, "\tau_{\boldsymbol{\mathfrak{A}}}"'] & T\boldsymbol{\mathfrak{B}} \arrow[l, "T\boldsymbol{k}_e"'] \arrow[d, "\tau_{\boldsymbol{\mathfrak{B}}}"] \\
\boldsymbol{X} \arrow[d, "k"'] & \tilde{\boldsymbol{X}} \arrow[l, "\boldsymbol{k}_e"'] \arrow[d, "\tilde{k}"] \\
X & \tilde{X} \arrow[l, "k_e"']
\end{tikzcd}$$

This embedding is *aligned* if

$$
\begin{aligned}
T\boldsymbol{k}_e\left(E^s_{\boldsymbol{X}}\right) &= E^s_{\tilde{\boldsymbol{X}}}, \quad \text{and} \\
T\boldsymbol{k}_e\left(E^u_{\boldsymbol{X}}\right) &= E^u_{\tilde{\boldsymbol{X}}}.
\end{aligned}
$$

The proof of lemma 9.3 can also be used to show that the only embeddings of hyperbolic pseudo-orbits are aligned embeddings.

Dually, if $\mathfrak{B}$ is a factor of $\mathfrak{A}$ as pseudo-orbits of $\boldsymbol{M}$ and f via the surjective functions $k_f : X \to \tilde{X}$, and $\boldsymbol{k}_f : \boldsymbol{X} \to \tilde{\boldsymbol{X}}$, then we can define a surjective function $T\boldsymbol{k}_f : T\mathfrak{A} \to T\mathfrak{B}$ as follows. For each $(n, x) \in \boldsymbol{X}$ let $(m, y) = \boldsymbol{k}_f(n, x)$ and note that $y = k_f(x)$. Define

$$T_{(n,x)}\boldsymbol{k}_f = \tilde{\boldsymbol{i}}^{-}T_{\boldsymbol{i}(n,x)}Id_{(m,n)}.$$

For $(n, x, v) \in T\mathfrak{A}$ this definition means that

$$T_{(n,x)}\boldsymbol{k}_f(n, x, v) = (m, y, v).$$

Note that this definition of $T\boldsymbol{k}_f$ makes the following pair of diagrams commute

$$
\begin{tikzcd}
T\mathfrak{A} \arrow[r, "\boldsymbol{F}"] \arrow[d, "T\boldsymbol{k}_f"'] & T\mathfrak{A} \arrow[dr, "\boldsymbol{i}"] \arrow[d, "T\boldsymbol{k}_f"'] & \\
& & TM \\
T\mathfrak{B} \arrow[r, "\tilde{\boldsymbol{F}}"] & T\mathfrak{B} \arrow[ur, "\tilde{\boldsymbol{i}}"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
T\mathfrak{A} \arrow[r, "T\boldsymbol{k}_f"] \arrow[d, "\pi_{\mathfrak{A}}"'] & T\mathfrak{B} \arrow[d, "\pi_{\mathfrak{B}}"'] \\
\boldsymbol{X} \arrow[r, "\boldsymbol{k}_f"] \arrow[d, "k"'] & \tilde{\boldsymbol{X}} \arrow[d, "\tilde{k}"'] \\
X \arrow[r, "k_f"] & \tilde{X}
\end{tikzcd}
$$

This factoring is *aligned* if

$$
\begin{aligned}
T\pi_M\left(E^s_{\boldsymbol{X}}\right) &= E^s_{\tilde{\boldsymbol{X}}}, \quad \text{and} \\
T\pi_M\left(E^u_{\boldsymbol{X}}\right) &= E^u_{\tilde{\boldsymbol{X}}}.
\end{aligned}
$$

Again, the proof of lemma 9.3 can also be used to show that the only factors of hyperbolic pseudo-orbits are aligned factors.

With these definitions, any hyperbolic pseudo-orbit, $\mathfrak{A}$, of M and f can be trivially identified with one or more hyperbolic pseudo-orbits $\boldsymbol{\mathfrak{A}}$ of $\boldsymbol{M}$ and f. It is important to note that a κ-λ-hyperbolic invariant set such as the Pesin set, *is* a κ-λ-hyperbolic 0-pseudo orbit. This implies, in particular, that all of the results in this thesis, can be directly applied to suitable subsets of the Pesin set.

9.2.6 Pseudo-hyperbolic pseudo-orbits

Our definitions of hyperbolic pseudo-orbits, given in the previous section, are not yet the most general definition with which we can prove the Shadowing lemma. In a numerical simulation, any estimate of hyperbolicity of a pseudo-orbit, i.e. a numerically simulated orbit, would *not*, in general, be made relative to an F-invariant splitting of the tangent spaces. At best, it will be made relative to a "pseudo-splitting", that is, a splitting for which the image, relative to F, of the splitting of the fibre $T_x\mathfrak{A}$ might be close to, but need not be equal to, the corresponding splitting of the fibre $T_{h(x)}\mathfrak{A}$.

In order to define pseudo-hyperbolicity of pseudo-orbits we use the notation,

$$\boldsymbol{E}_x^{s+} = F_{h(x)}^{-1}\boldsymbol{E}_{h(x)}^{s}, \qquad \boldsymbol{E}_x^{u+} = F_{h(x)}^{-1}\boldsymbol{E}_{h(x)}^{u},$$
$$\boldsymbol{E}_x^{s-} = F_{h^{-1}(x)}\boldsymbol{E}_{h^{-1}(x)}^{s}, \qquad \boldsymbol{E}_x^{u-} = F_{h^{-1}(x)}\boldsymbol{E}_{h^{-1}(x)}^{u}.$$

That is the splitting $\boldsymbol{E}_x^{s+} \oplus \boldsymbol{E}_x^{u+}$ is the image via $F_{h(x)}^{-1}$ of the splitting in the fibre of $h(x)$. Similarly, the splitting $\boldsymbol{E}_x^{s-} \oplus \boldsymbol{E}_x^{u-}$ is the image via $F_{h^{-1}(x)}$ of the splitting in the fibre of $h^{-1}(x)$.

If there exist uniform bounds, K_+ and K_-, on the fibre operator norms of the fibre bundle maps F and F^{-1} respectively, then if the splitting $\boldsymbol{E}_x^{s} \oplus \boldsymbol{E}_x^{u}$ is h_n-non-degenerate then the splittings $\boldsymbol{E}_{h^{-1}(x)}^{s+} \oplus \boldsymbol{E}_{h^{-1}(x)}^{u+}$ and $\boldsymbol{E}_{h(x)}^{s-} \oplus \boldsymbol{E}_{h(x)}^{u-}$ are $K_- h_n$ and $K_+ h_n$-non-degenerate respectively. Since M is compact and $F_{x,y}$ is continuous, we can define the finite positive constants $K_\pm = \sup_{y,x \in M}\{\|F_{x,y}\|\}$. These constants bound the fibre operator norms of the fibre bundle maps F and F^{-1} for any pseudo-orbit of M and f. These constants will also bound the fibre operator norms of the fibre bundle maps $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ for any pseudo-orbit of $\boldsymbol{M}_o$ and f.

Theorem 10.1 asserts the existence of constants K_+ and K_- which similarly bound the fibre operator norms of the fibre bundle maps $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ for any pseudo-orbit of $\boldsymbol{M}$ and f *with respect to* any *adapted metric*. These considerations imply that we can always find constants K_+ and K_- which bound the fibre operator norms of $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ for any pseudo-orbit of $\boldsymbol{M}$ and f with respect to any metric we will use in the remainder of this thesis. Note that the value of these constants depend crucially on the pseudo-orbit and the Riemannian metric used.

We will also define

$$F_x^s = p_{h(x)}^s F_x, \quad F_x^{s,0} = p_x^s, \quad F_x^{s,n} = F_{h^n(x)}^s \circ \cdots \circ F_x^s,$$
$$F_x^u = p_{h^{-1}(x)}^u F_x^{-1}, \quad F_x^{u,0} = p_x^u, \quad F_x^{u,n} = F_{h^{-n}(x)}^u \circ \cdots \circ F_x^u.$$

Again, we define a (factored) pseudo-hyperbolic pseudo-orbit via a number of intermediate subdefinitions. We begin by defining a pseudo-hyperbolic pseudo-orbit $\mathfrak{A}$. A classified pseudo-orbit, $\mathfrak{A}$ of M and f, is (C_*^0) (δ_n)*-κ-λ-pseudo hyperbolic with h_0-hyperbolic blocks* for a non-negative sequence (δ_n), $0 < \lambda < \frac{1}{\kappa} \leq 1$ and $1 \leq h_0$, if there exists a splitting $T\mathfrak{A} = \boldsymbol{E}_X^s \oplus_X \boldsymbol{E}_X^u$, for which

$$
\begin{aligned}
|F_x^{s,m}(v_s)| &\leq h_n \lambda^m \kappa^m |v_s| && (9.4a)\\
|F_x^{u,m}(v_u)| &\leq h_n \lambda^m \kappa^m |v_u| && (9.4b)\\
\max\{\|p_x^s\|, \|p_x^u\|\} &\leq h_n && (9.4c)\\
d\left(\boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u, \boldsymbol{E}_x^{s+} \oplus \boldsymbol{E}_x^{u+}\right) &\leq \delta_n && (9.4d)\\
d\left(\boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u, \boldsymbol{E}_x^{s-} \oplus \boldsymbol{E}_x^{u-}\right) &\leq \delta_n && (9.4e)
\end{aligned}
$$

for all $m, n \geq 0$, $x \in X_n$, $v_s \in \boldsymbol{E}_x^s$ and $v_u \in \boldsymbol{E}_x^u$. Furthermore, we assume that for each $0 \leq n$ the n^{th} hyperbolic block, P_n, is a closed subset of X, and for each P_n the functions $\boldsymbol{E}^s$, $\boldsymbol{E}^u : P_n \to G_{i(P_n)}\mathfrak{A}$, $\boldsymbol{E}^s \oplus_{P_n} \boldsymbol{E}^u : P_n \to S_{h_n}(T_{P_n}\mathfrak{A})$, and $F : T_{P_n}\mathfrak{A} \to T_{P_{n+1}}\mathfrak{A}$ are each C^0.

Note that in conditions 9.4d and 9.4e, the quantity, $d\left(\boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u, \boldsymbol{E}_x^{s\pm} \oplus \boldsymbol{E}_x^{u\pm}\right)$, is the fibre distance between the pairs of splittings $\boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u$ and $\boldsymbol{E}_x^{s\pm} \oplus \boldsymbol{E}_x^{u\pm}$ in the bundle of $K_\pm h_{n+1}$-non-degenerate splittings, $S_{K_\pm h_{n+1}}(T_{P_n}\mathfrak{A})$ where $K_\pm$ are the constants, which depend on the pseudo-orbit and Riemannian metric, which were discussed above.

An unfactored pseudo-orbit, $\mathfrak{A}$ of $\boldsymbol{M}$ and f, is (C^0_*) (δ_n)-*κ-λ-pseudo hyperbolic with h_0-hyperbolic blocks* for a non-negative sequence (δ_n), $0 < \lambda < \frac{1}{\kappa} \leq 1$ and $1 \leq h_0$, if there exists a splitting $T\mathfrak{A} = \boldsymbol{E}^s_{\mathfrak{A}} \oplus_X \boldsymbol{E}^u_{\mathfrak{A}}$, and

$$
\begin{aligned}
\left|F^{s,m}_x(v_s)\right| &\leq h_n \lambda^m \kappa^m \left|v_s\right| && (9.5a)\\
\left|F^{u,m}_x(v_u)\right| &\leq h_n \lambda^m \kappa^m \left|v_u\right| && (9.5b)\\
\max\left\{\|p^s_x\|, \|p^u_x\|\right\} &\leq h_n && (9.5c)\\
d\left(\boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x, \boldsymbol{E}^{s+}_x \oplus \boldsymbol{E}^{u+}_x\right) &\leq \delta_n && (9.5d)\\
d\left(\boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x, \boldsymbol{E}^{s-}_x \oplus \boldsymbol{E}^{u-}_x\right) &\leq \delta_n && (9.5e)
\end{aligned}
$$

for all $m, n \geq 0$, $x \in X_n$, $v_s \in \boldsymbol{E}^s_x$ and $v_u \in \boldsymbol{E}^u_x$. Furthermore, we assume that for each n, the functions $\boldsymbol{E}^s$, $\boldsymbol{E}^u : X_n \to G_{i(X_n)}\mathfrak{A}$, $\boldsymbol{E}^s \oplus_{X_n} \boldsymbol{E}^u : X_n \to S_{h_n}(T_{X_n}\mathfrak{A})$, and $\boldsymbol{F} : T_{P_n}\mathfrak{A} \to T_{P_{n+1}}\mathfrak{A}$ are each C^0.

A factored pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, is (C^0_*) (δ_n)-*κ-λ-pseudo hyperbolic with h_0-hyperbolic blocks* for a non-negative sequence (δ_n), $0 < \lambda < \frac{1}{\kappa} \leq 1$, and $1 \leq h_0$, if

1. the factor pseudo-orbit, $\mathfrak{A}(X, h, i, \chi)$, is C^0_* (δ_n)-κ-λ-pseudo hyperbolic with h_0-hyperbolic blocks,

2. the unfactored pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, is C^0_* (δ_n)-κ-λ-pseudo hyperbolic with h_0-hyperbolic blocks,

Finally, a factored pseudo-hyperbolic pseudo-orbit, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$, of $\boldsymbol{M}$ and f, is *aligned* if

$$
\begin{aligned}
\boldsymbol{i}^* T\pi_M\left(\boldsymbol{E}^s_{\boldsymbol{X}}\right) &= E^s_X, \text{ and}\\
\boldsymbol{i}^* T\pi_M\left(\boldsymbol{E}^u_{\boldsymbol{X}}\right) &= E^u_X.
\end{aligned}
$$

Note that pseudo-hyperbolic pseudo-orbits need not be aligned. Indeed, the construction in the proof of lemma 12.1 does not construct an aligned pseudo-hyperbolic pseudo-orbit.

Consider a pair of factored pseudo-orbits, $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{f})$ and $\boldsymbol{\mathfrak{B}}\left(\mathfrak{B}, \bar{k}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{i}}, \bar{\boldsymbol{\chi}}, \bar{\boldsymbol{f}}\right)$. Recall that, in the previous section, we defined the map $T\boldsymbol{k}_f : \boldsymbol{X} \to \bar{\boldsymbol{X}}$ $\left(T\boldsymbol{k}_s : \bar{\boldsymbol{X}} \to \boldsymbol{X}\right)$ corresponding to the surjective (injective) factor (embedding) map k_f (k_e) of $\mathfrak{B}$ with $\mathfrak{A}$. We can use this map along with the techniques used in the proof of lemma 9.3 to show that if a (δ_n)-κ-λ-pseudo hyperbolic pseudo-orbit $\mathfrak{B}$ is a monotonic factor of a pseudo-orbit $\mathfrak{A}$, then $\mathfrak{A}$ is itself a (δ_n)-κ-λ-pseudo hyperbolic pseudo-orbit. Unfortunately, a similar statement can not be made about monotonic embeddings.

Note that this definition of a (δ_n)-κ-λ-pseudo hyperbolic (α_n)-pseudo orbit encompasses all of our previous definitions. That is, a 0-κ-λ-hyperbolic pseudo-orbit is a hyperbolic pseudo-orbit, a κ-λ-hyperbolic 0-pseudo-orbit is a (weakly) hyperbolic invariant set, and a 1-λ-hyperbolic invariant set is a uniformly hyperbolic invariant set.

Chapter 10

Adapted metrics for pseudo-hyperbolic pseudo-orbits

In this chapter we are interested in showing that corresponding to any appropriate pseudo-hyperbolic pseudo-orbit, $\mathfrak{A}$, of $\boldsymbol{M}$ and f, there exists an "adapted" C^∞ Riemannian metric on the classifying manifold, $\boldsymbol{M}$, with respect to which the eventual contraction of the pseudo-orbit $\mathfrak{A}$ with respect to the original metric becomes immediate contraction with respect to the adapted metric.

10.1 Adapted metrics

Given an invariant set Λ of a Riemannian manifold M which is C-uniformly λ-hyperbolic, Mather [Mat68][1] has shown that for any $\lambda < \check{\lambda} < 1 < \check{C}$ there exists at least one Riemannian metric, which is C^∞ on the whole of M, for which the invariant set Λ is $\check{C}$-uniformly $\check{\lambda}$-hyperbolic with respect to the adapted metric. Since the splitting $T_\Lambda M = \boldsymbol{E}^s_\Lambda \oplus_\Lambda \boldsymbol{E}^u_\Lambda$ can be made arbitrarily close to being perpendicular with respect to the adapted metric, conditions 9.1a, 9.1b and 9.1c are satisfied with $C = \check{C}$ arbitrarily close to 1. This means that the eventual contraction (expansion) with respect to the original metric can be changed to immediate contraction (expansion) with respect to the adapted metric.

Just as for uniformly hyperbolic sets, Pesin [Pes76, FHY83, PS89] has shown that there is at least one metric associated to any κ-λ-hyperbolic invariant set

[1] See also [Kat81].

with respect to which the eventual contraction (expansion) with respect to the original metric is changed to immediate contraction (expansion) with respect to Pesin's adapted "metric". Unfortunately, in this case, the metric is only defined in the tangent bundle of the invariant set itself, and moreover the metric is *not* C^∞ nor even C^0. Pesin was, however, able to use this metric to prove a Stable Manifold theorem for κ-λ-hyperbolic sets.

Combining the techniques of *both* Mather and Pesin, we will show, in the next theorem, that given any C^∞ Riemannian metric of $\boldsymbol{M}$ and for any

$$0 \leq \bar{\delta}, \quad 0 < \lambda\kappa < \bar{\lambda} < 1 \leq \kappa, \quad 1 \leq h_0, \text{ and } 1 < \bar{h}_0$$

there exists a κ^4-slowly decreasing sequence (δ_n) for which for any (δ_n)-κ-λ-pseudo hyperbolic pseudo-orbit, $\mathfrak{A}$ of $\boldsymbol{M}$, which has h_0-hyperbolic blocks, there exists a C^∞ Riemannian metric of the classifying manifold $\boldsymbol{M}$ with respect to which $\mathfrak{A}$ is a $\bar{\delta}$-1-$\bar{\lambda}$-pseudo hyperbolic pseudo-orbit with $\bar{h}_0$-hyperbolic blocks. We call this metric of $\boldsymbol{M}$ the *adapted metric* of $\boldsymbol{M}$ with respect to the pseudo-hyperbolic pseudo-orbit $\mathfrak{A}$.

An important property of the adapted metric of $\boldsymbol{M}$ is that while the topological equivalence between the M_n with the adapted metric and M_n with the original metric gets progressively poorer, it does so only κ^2-slowly. We will make repeated use of this slow change in the equivalence relationship to translate results proven relative to the adapted metric back to results for the original metric.

Theorem 10.1 (Adapted metrics) *Consider*

$$0 \leq \bar{\delta}, \quad 0 < \lambda\kappa < \bar{\lambda} < 1 \leq \kappa, \quad 1 \leq h_0, \text{ and } 1 < \bar{h}_0,$$

let $\rho = \frac{\lambda\kappa}{\bar{\lambda}}$, *and let* $\delta_n \leq \frac{\bar{\delta}}{2h_0^2}\sqrt{1-\rho^2}$ *and note that* (δ_n) *is a* κ^4*-slowly decreasing sequence. Finally, assume that* $\mathfrak{A}$ *is an aligned factored* (δ_n)-κ-λ*-pseudo hyperbolic pseudo-orbit of* $\boldsymbol{M}$ *with* h_0*-hyperbolic blocks with respect to the original* C^∞ *Riemannian metric of* $\boldsymbol{M}$.

Then there exists a C^∞ *Riemannian metric of* $\boldsymbol{M}$ *with respect to which* both *the pseudo-orbit* $\mathfrak{A}$ and *the pseudo-orbit's maximally shifted closure,* $\bar{\mathfrak{A}}$*, are* $\bar{\delta}$-1-$\bar{\lambda}$*-pseudo hyperbolic pseudo-orbits of* $\boldsymbol{M}$ *and* f *with* $\bar{h}_0$*-hyperbolic blocks. We* *call this metric the adapted metric of* $\boldsymbol{M}$ *for the pseudo-orbit* $\mathfrak{A}$. *We denote this metric and its associated norm by* $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ *and* $[\![\cdot]\!]$ *respectively.*

There exists a κ^2*-slowly increasing sequence,* $B_n = h_n^2\sqrt{\frac{1}{1-\rho^2}}$, *such that,* $\boldsymbol{M}$ *with the adapted metric is* $\frac{1}{\sqrt{\eta}}$*-*(B_n)*-related to* $\boldsymbol{M}$ *with the original metric. Moreover, there is a neighbourhood* U *of* $\boldsymbol{i}(\boldsymbol{X})$ *in* M *for which if* $p \notin U$ *then* $[\![v]\!] = |v|$ *for all* $v \in T_p\boldsymbol{M}$.

If $\mathfrak{A}$ *is an invariant set, there exist constants* K_+ *and* K_- *which only depend on* $\tilde{\lambda}$, h_0, $\tilde{\delta}$, $\sup_{x\in\boldsymbol{X}}\{\|F_x\|\}$, *and* $\sup_{x\in\boldsymbol{X}}\{\|F_x^{-1}\|\}$, *such that we have*

$$\|[\![F_x]\!]\| \le K_+, \quad \text{and,} \quad \left\|\left[\!\left[F_x^{-1}\right]\!\right]\right\| \le K_-,$$

for all $x \in \boldsymbol{X}$.

Moreover, if $x \in P_n \subset X$. *then* $\left\|\left[\!\left[Id_{(m,m-1)}\right]\!\right]\right\|_{(m,x;m-1,x)} \le 1$ *for all* $m > n$.

Proof: This is the most tedious and difficult proof in this part of the thesis. Note that the statement of this theorem specifically allows $\kappa = 1$. In this case the κ^2-slowly increasing sequence, (B_n), is a constant sequence.

While we have chosen to state the theorem in terms of a single Riemannian metric of $\boldsymbol{M}$, it is more natural to prove the theorem in terms of a countable collection of Riemannian metrics, one on each copy, M_n, of M in $\boldsymbol{M}$. In order to distinguish the various metrics contained in this countable collection of metrics, we will denote the metric and norm corresponding to M_n by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_n$, and $[\![\cdot]\!]_n$ respectively. Furthermore, we will use the notation, $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{(n,x)}$, and $[\![\cdot]\!]_{(n,x)}$, to stress in which tangent space the inner product or norm is taken. With this notation, in order to prove this theorem, we must prove

1. For every $0 \le n$ there exists a C^∞ Riemannian metric, $\langle\!\langle\cdot,\cdot\rangle\!\rangle_n$, on M_n. This Riemannian metric induces a norm, $[\![\cdot]\!]_n$, on $T_{M_n}\boldsymbol{M}$, and a distance metric, $d\langle\!\langle\cdot,\cdot\rangle\!\rangle_n$, on $S_2(TM_n)$. With respect to these objects we have that for all $x \in \boldsymbol{i}(\boldsymbol{X}_n) \subset M_n$

$$\begin{aligned}
[\![\boldsymbol{F}_x^n(v_s)]\!]_{m_+} &\le \tilde{\lambda}\,[\![v_s]\!]_n \\
[\![\boldsymbol{F}_x^n(v_u)]\!]_{m_-} &\le \tilde{\lambda}\,[\![v_u]\!]_n \\
\max\{\|[\![p_x^s]\!]\|_n, \|[\![p_x^u]\!]\|_n\} &\le \tilde{h}_0 \\
d\left(\left(\boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u, \boldsymbol{E}_x^{s*} \oplus \boldsymbol{E}_x^{u*}\right)\right)_n &\le \tilde{\delta}
\end{aligned}$$

where $\mathbf{h}(x) \in \boldsymbol{X}_{m_+}$, $\mathbf{h}^{-1}(x) \in \boldsymbol{X}_{m_-}$, $v_s \in \boldsymbol{E}^s_x$, and $v_u \in \boldsymbol{E}^u_x$.

2. There exists a κ^2-slowly increasing sequence, (B_n), for which for each n, the adapted norm, $[\cdot]_n$, on M_n is related to the original norm by

$$\frac{1}{\sqrt{2}}|v| \leq [v]_n \leq B_n |v| \tag{10.1}$$

for all $v \in T_x\boldsymbol{M}$ and all $x \in M_n$.

3. If $\mathfrak{A}$ is an invariant set, there exist constants K_+, and K_- for which for all n, $x \in X_n$, and $v \in T_x\boldsymbol{M}$ we have

$$[F_x(v)]_{m_+} \leq K_+ [v]_n, \quad \text{and} \quad , \left\|\left[F_x^{-1}(v)\right]\right\|_{m_-} \leq K_- [v]_n.$$

4. If $x \in P_n$ then $[v]_{(m-1,x)} \leq [v]_{(m,x)}$ for all $m > n$, and $v \in T_xM$.

There are two main steps in the proof of this theorem. In the first step we use the *classified pseudo-orbit*, $\mathfrak{A}(X, h, i, \chi)$, *of M and f* and the original metric of M_n to define, for each n, a pre-adapted inner product defined in the tangent bundle, TM, over the n^{th} hyperbolic block, P_n viewed as points in M. We then show that these pre-adapted metrics satisfy the required inequalities when restricted to $T_{P_n}M$. In the second step, we show that we can take C^∞ approximations to the pre-adapted metrics in the neighbourhoods U_n which preserve all of our inequalities. Using partitions of unity, we can "glue" these approximations in the neighbourhoods U_n to the original metric of M_n outside of the U_n, to obtain a C^∞ metric defined on the whole of each M_n.

Since the given pseudo-hyperbolic pseudo-orbit was assumed to be both be both factored and aligned this countable collection of Riemannian metrics which we have adapted to the factored pseudo-orbit, $\mathfrak{A}$, is, as *a* metric of $\boldsymbol{M}$, adapted for $\mathfrak{A}$ as well. Since for all n, $P_n \subset P_{n+1}$ and moreover, since we have consistently constructed each metric of M_n to be adapted for all of the points in P_n (as opposed to all of the points in X_n), the adapted metric of $\boldsymbol{M}$ is also adapted to the maximally shifted closure, $\bar{\mathfrak{A}}$ of $\mathfrak{A}$.

To simplify the notation we will identify the point $x \in \boldsymbol{X}$ ($x \in X$) with its image $\mathbf{i}(x) \in \boldsymbol{M}$ ($i(x) \in M$) via the index injection $\mathbf{i}$ (i).

10.1.1 Step 1: The pre-adapted metric for $P_n \subset M_n$

The pre-adapted metric is constructed in such a way that the stable and unstable subspaces are perpendicular. We begin by constructing a pre-adapted metric in the stable directions. To do this, fix a positive n, and consider $x \in P_n \subset X$. For $v_s, \tilde{v}_s \in E_x^s$ define

$$\langle\langle v_s, \tilde{v}_s \rangle\rangle_{(n,x)} = \sum_{i=0}^{N_n} \tilde{\lambda}^{-2i} \left\langle F_x^{s,i}(v_s), F_x^{s,i}(\tilde{v}_s) \right\rangle_{h^s(x)}$$

where $N_n = N_0 + n\Delta N$. The fixed constants N_0 and ΔN will be determined later. If $\kappa = 1$ then the constant ΔN will be chosen to be zero. Let $|v_s|_x^2 = \langle v_s, v_s \rangle_x$, and $[v_s]_{(n,x)}^2 = \langle\langle v_s, v_s \rangle\rangle_{(n,x)}$. We note that this definition implies that $[v_s]_{(n-1,x)} \leq [v_s]_{(n,x)} \leq [v_s]_{(n+1,x)}$ and hence it is enough to prove inequalities 10.3 and 10.8a, with respect to $[F_x^s(v_s)]_{n+1}$.

For the following x is in P_n, $f(x)$ is in P_{n+1} and v_s is in E_x^s. Consider

$$\begin{aligned} [F_x^s(v_s)]_{n+1}^2 &= \sum_{i=0}^{N_{n+1}} \tilde{\lambda}^{-2i} \left| F_x^{s,i+1}(v_s) \right|^2 \\ &\leq \sum_{i=N_n+1}^{N_{n+1}} \tilde{\lambda}^{-2i} \left| F_x^{s,i+1}(v_s) \right|^2 + \tilde{\lambda}^2 [v_s]_n^2 - \tilde{\lambda}^2 |v_s|^2. \end{aligned}$$

We begin by defining some constants to simplify the notation. Let $\rho = \frac{\lambda\kappa}{\tilde{\lambda}} < 1$. Let $\rho_{\Delta N}^2 = \sum_{i=1}^{\Delta N} \rho^{2i}$ and $\rho_{\Delta N}^2 = 0$ if $\Delta N = 0$. Recall that for $x \in P_n$, $h_n = h_0 \kappa^n$ and $|F_x^{s,m}(v_s)| \leq h_n \lambda^m \kappa^m |v_s|$. This last inequality allows us to estimate

$$\begin{aligned} \sum_{i=N_n+1}^{N_{n+1}} \tilde{\lambda}^{-2i} \left| F_x^{s,i+1}(v_s) \right|^2 &\leq \tilde{\lambda}^2 \rho^{2N_n} h_0^2 \kappa^{2n} \left[\sum_{i=1}^{N_{n+1}-N_n} \rho^{2i} \right] |v_s|^2 \\ &\leq \tilde{\lambda}^2 \left[\rho^{2N_0} h_0^2 \rho_{\Delta N}^2 \right] \left[\rho^{\Delta N} \kappa \right]^{2n} |v_s|^2. \end{aligned} \tag{10.2a}$$

Hence

$$[F_x^s(v_s)]_{n+1}^2 \leq \tilde{\lambda}^2 [v_s]_n^2 - \tilde{\lambda}^2 |v_s|^2 \left(1 - \left[\rho^{2N_0} h_0^2 \rho_{\Delta N}^2 \right] \left[\rho^{\Delta N} \kappa \right]^{2n} \right)$$

But

$$\begin{aligned} [v_s]_n^2 &= \sum_{i=0}^{N_n} \tilde{\lambda}^{-2i} \left| F_x^{s,i}(v_s) \right|^2 \\ &\leq h_n^2 \left[\sum_{i=0}^{N_n} \rho^{2i} \right] |v_s|^2 \\ &\leq \left[\frac{h_n^2}{1-\rho^2} \right] |v_s|^2 \end{aligned}$$

Hence

$$[F_x^s(v_s)]_{n+1}^2 \le \tilde{\lambda}^2 \left(1 - \frac{1 - \left[\rho^{2N_0} h_0^2 \rho_{\triangle N}^2\right] \left[\rho^{\triangle N} \kappa\right]^{2n}}{h_n^2/(1-\rho^2)} \right) [v_s]_n^2$$

If $\triangle N$ is chosen so that $\rho^{\triangle N}\kappa \le 1$ (if $\kappa = 1$ then choose $\triangle N = 0$), and if N_0 is chosen so that $\rho^{2N_0} h_0^2 \rho_{\triangle N}^2 < 1$, then for all $n \ge 0$ we will have

$$[F_x^s(v_s)]_{n+1}^2 \le \tilde{\lambda}^2 [v_s]_n^2 \tag{10.3}$$

So far we have constructed the pre-adapted metric in the stable directions, by using essentially the same construction using F_x^u and h^{-1} instead of F_x^s and h, we can construct the pre-adapted metric in the unstable directions. To do this we define

$$\langle\!\langle v_u, \tilde{v}_u \rangle\!\rangle_{(n,x)} = \sum_{i=0}^{N_n} \tilde{\lambda}^{-2i} \left\langle F_x^{u,i}(v_u), F_x^{u,i}(\tilde{v}_u) \right\rangle_{h_n^{-}}$$

for all $v_u, \tilde{v}_u \in E_x^u$ for any x in P_n. Again we note that this definition implies that $[v_u]_{n-1} \le [v_u]_n \le [v_u]_{n+1}$. Exactly the same estimates now hold for F_x^u, hence we have, for all $n \ge 1$,

$$[F_x^u(v_u)]_{n+1}^2 \le \tilde{\lambda}^2 [v_u]_n^2 . \tag{10.4}$$

Now consider $v, \tilde{v} \in T_xM$ for some $x \in P_n$. Let $v = v_s + v_u$ and $\tilde{v} = \tilde{v}_s + \tilde{v}_u$ where $v_s, \tilde{v}_s \in \tilde{E}_x^s$ and $v_u, \tilde{v}_u \in \tilde{E}_x^u$. We define

$$\begin{aligned} \langle\!\langle v_s, v_u \rangle\!\rangle_{(n,x)} &= 0, \quad \text{and} \\ \langle\!\langle v, \tilde{v} \rangle\!\rangle_{(n,x)} &= \langle\!\langle v_s, \tilde{v}_s \rangle\!\rangle_{(n,x)} + \langle\!\langle v_u, \tilde{v}_u \rangle\!\rangle_{(n,x)} . \end{aligned}$$

and $[v]_{(n,x)}^2 = \langle\!\langle v, v \rangle\!\rangle_{(n,x)}$. In particular these definitions imply that

$$[v]_n \le [v]_{n+1}, \quad \text{and} \tag{10.5a}$$

$$\max\left\{ \|[p_x^s]\|_{(n,x)}, \|[p_x^s]\|_{(n,x)} \right\} = 1 \tag{10.5b}$$

for all $x \in P_n \subset P_{n+1} \subset X$ and all $v \in T_xM$.

Comparing the norms We are now interested in the relationship between the original Riemannian norm, $|\cdot|$, and the n^{th} pre-adapted norm, $[\cdot]_n$. We consider this relationship for $x \in P_n$. Consider $v \in T_xM$ where $v = v_s + v_u$.

For the estimate from below, from the definition of $[v_s]_n$ we know that $|v_s| \leq [v_s]_n$. Similarly we also have $|v_u| \leq [v_u]_n$. Since $(a+b)^2 \leq 2(a^2+b^2)$ we have

$$|v| \leq |v_s| + |v_u| \leq [v_s]_n + [v_u]_n \leq \sqrt{2}\,[v]_n\,. \tag{10.6}$$

For the estimate from above, from estimates made above, we know that

$$[v_s]_n \leq \left[\frac{h_n^2}{1-\rho^2}\right] |v_s|^2\,.$$

Similar arguments can be use to show that

$$[v_u]_n \leq \left[\frac{h_n^2}{1-\rho^2}\right] |v_u|^2\,.$$

Since $\mathfrak{A}$ has h_0-hyperbolic blocks, we know that $\max\{\|p_x^s\|, \|p_x^u\|\} \leq h_n$. Hence

$$|v_s|^2 + |v_u|^2 = |p_x^s(v)|^2 + |p_x^u(v)|^2 \leq 2h_n^2\,|v|^2\,.$$

This means that we have the following estimate for the upper bound

$$[v]_n^2 = [v_s]_n^2 + [v_u]_n^2 \leq \left[\frac{h_n^2}{1-\rho^2}\right]\left(|v_s|^2 + |v_u|^2\right) \leq \left[\frac{2h_n^4}{1-\rho^2}\right]|v|^2\,. \tag{10.7}$$

Define $B_n = h_n^2\sqrt{\frac{2}{1-\rho^2}}$ and note that B_n is a κ^2-slowly *increasing* sequence.

Controlling the pseudo-hyperbolicity We now want to show that

$$d\left(\left(E_x^s \oplus E_x^u, E_x^{s\pm} \oplus E_x^{u\pm}\right)\right)_n \leq \tilde{\delta}.$$

We start by showing that $[p_x^u p_x^{s\pm}]_n \leq \tilde{\delta}$. To do this note that, since $p_x^u(T_xM) = E_x^u$, we have

$$\begin{aligned}
\left\|p_x^u p_x^{s\pm}\right\|_n^2 &\leq \sum_{i=0}^{N_n} \tilde{\lambda}^{-2i}\left\|F_x^{u,i} p_x^u p_x^{s\pm}\right\|^2 \\
&\leq \sum_{i=0}^{N_n} h_n^2 \rho^{2i}\left\|p_x^u p_x^{s\pm}\right\|^2 \\
&\leq h_n^2\left[\frac{1}{1-\rho^2}\right]\left\|p_x^u p_x^{s\pm}\right\|^2 \\
&\leq \left[\frac{h_n^2}{1-\rho^2}\right]\delta_n^2 \\
&\leq \frac{\tilde{\delta}^2}{4h_n^2} \\
&\leq \tilde{\delta}^2.
\end{aligned}$$

Similar arguments show that $[p_x^s p_x^{u\pm}]_n \leq \tilde{\delta}$.

To show that $[p_x^{u\pm} p_x^s]_n \leq \tilde{\delta}$ consider

$$\begin{aligned} [\![p_x^u p_x^{u\pm} p_x^s]\!]_n^2 &\leq h_n^2 \left[\frac{1}{1-\rho^2}\right] \left\| p_x^u p_x^{u\pm} p_x^s \right\|^2 \\ &\leq h_n^4 \left[\frac{1}{1-\rho^2}\right] \left\| p_x^{u\pm} p_x^s \right\|^2 \\ &\leq \left[\frac{h_n^4}{1-\rho^2}\right] \delta_n^2 \\ &\leq \frac{\tilde{\delta}^2}{4}. \end{aligned}$$

Again, similar arguments show that $[p_x^s p_x^{u\pm} p_x^s]_n \leq \frac{\tilde{\delta}}{2}$. This in turn implies that

$$[\![p_x^{u\pm} p_x^s]\!]_n \leq [\![p_x^u p_x^{u\pm} p_x^s]\!]_n + [\![p_x^s p_x^{u\pm} p_x^s]\!]_n \leq \tilde{\delta}.$$

The arguments required to show that $[p_x^{s\pm} p_x^u]_n \leq \tilde{\delta}$ are similar.

Estimating F_x^u and F_x^s in $T\mathfrak{M}$: We have already obtained estimates for the norms of $F_x^s|_{E_x^s}$ and $F_x^u|_{E_x^u}$. We are now interested in establishing estimates for these norms over the whole of each fibre of $T\mathfrak{M}$.

To do this it will be useful to obtain bounds for the quantities like $[p_x^u p_x^{u\pm}]_n$. To do this note that

$$\begin{aligned} [\![p_x^u p_x^{u\pm}]\!]_n &= [\![p_x^u \left(Id - p_x^{s\pm} \right)]\!]_n \\ &\leq [p_x^u]_n + [\![p_x^u p_x^{s\pm}]\!]_n \\ &\leq 1 + \tilde{\delta}. \end{aligned}$$

Similar arguments show that

$$[\![p_x^{u\pm} p_x^u]\!]_n , [\![p_x^s p_x^{s\pm}]\!]_n , [\![p_x^{s\pm} p_x^s]\!]_n \leq 1 + \tilde{\delta}.$$

Now consider $x \in P_n$ and $v \in E_x^u$. Note that $p_{h(x)}^u F_x(v) \in E_{h(x)}^u$, that $F_{h(x)}^u p_{h(x)}^u F_x = p_x^u F_{h(x)}^{-1} p_{h(x)}^u F_x = p_x^u p_x^{u+}$, that the arguments used to prove inequality 10.2a show that $\sum_{i=N_n+1}^{N_{n+1}} \tilde{\lambda}^{-2i} \left| F_x^{u,i} \left(p_x^u p_x^{u+} v \right) \right| \leq \left| p_x^u p_x^{u+} v \right|$, and finally, that $|v|^2 \leq 2\,[v]_n^2$. If we let $\|F\| = \sup_{x \in X} \{F_x\}$ then these facts together imply that

$$[\![p_{h(x)}^u F_x(v)]\!]_{n+1}^2$$

$$
\begin{aligned}
&= \sum_{i=0}^{N_{n+1}} \bar{\lambda}^{-2i} \left| F_{h(x)}^{u,i} \left(p_{h(x)}^{u} F_x (v) \right) \right|^2 \\
&\leq \bar{\lambda}^{-2(N_{n+1}+1)} \left| F_{h(x)}^{u,(N_{n+1}+1)} \left(p_{h(x)}^{u} F_x (v) \right) \right|^2 \\
&\quad + \sum_{i=1}^{N_{n+1}} \bar{\lambda}^{-2i} \left| F_x^{u,i-1} \left(p_x^u p_x^{u+} v \right) \right|^2 + \left| p_{h(x)}^{u} F_x (v) \right|^2 \\
&\leq \bar{\lambda}^{-2} \left(\sum_{i=N_n+1}^{N_{n+1}} \bar{\lambda}^{-2i} \left| F_x^{u,i} \left(p_x^u p_x^{u+} v \right) \right|^2 + \left\| p_x^u p_x^{u+} v \right\|_n^2 \right) + \left| F_x p_x^{u+} v \right|^2 \\
&\leq \bar{\lambda}^{-2} \left(\left| p_x^u p_x^{u+} v \right|^2 + \left\| p_x^u p_x^{u+} v \right\|_n^2 \right) + \|F\|^2 \left(\left| p_x^{u+} p_x^u v \right|^2 + \left| p_x^{u+} p_x^s v \right|^2 \right) \\
&\leq 3\bar{\lambda}^{-2} \left\| p_x^u p_x^{u+} v \right\|_n^2 + \|F\|^2 \left(2 \left\| p_x^{u+} p_x^u v \right\|_n^2 + 2 \left\| p_x^{u+} p_x^s v \right\|_n^2 \right) \\
&\leq \left(\frac{3}{\bar{\lambda}^2} + 4 \|F\|^2 \right) \left(1 + \bar{\delta} \right) [v]_n^2 .
\end{aligned}
$$

We will also need to consider $\left\| p_{h(x)}^s F_x p_x^u v \right\|_{n+1}^2$. To do this note that $F_x \circ p_x^{s+} (T_x M) \subset E_x^s$ and that $\left| F_{h(x)}^{s,m} (v_s) \right| \leq h_{n+1} \lambda^m \kappa^m |v_s|$. This means that

$$
\begin{aligned}
\left\| p_{h(x)}^s F_x p_x^u v \right\|_{n+1}^2 &= \sum_{i=0}^{N_{n+1}} \bar{\lambda}^{-2i} \left| p_{h(x)}^s F_x p_x^u v \right|^2 \\
&\leq \sum_{i=0}^{N_{n+1}} h_{n+1}^2 \rho^{2i} \left| F_x p_x^{s+} p_x^u v \right|^2 \\
&\leq h_{n+1}^2 \left[\frac{1}{1-\rho^2} \right] \|F_x\|^2 \left\| p_x^{s+} p_x^u \right\|^2 |v|^2 \\
&\leq 2\bar{\delta}^2 \kappa^2 \|F\|^2 [v]_n^2 .
\end{aligned}
$$

Now consider

$$
\begin{aligned}
&[F_x v]_{n+1}^2 \\
&\quad \leq \left\| p_{h(x)}^s F_x p_x^s v \right\|_{n+1}^2 + \left\| p_{h(x)}^s F_x p_x^u v \right\|_{n+1}^2 + \left\| p_{h(x)}^u F_x (v) \right\|_{n+1}^2 \\
&\quad \leq \bar{\lambda}^2 [v_s]_n^2 + 2\bar{\delta}^2 \kappa^2 \|F\|^2 [v]_n^2 + \left(\frac{3}{\bar{\lambda}^2} + 4 \|F\|^2 \right) \left(1 + \bar{\delta} \right) [v]_n^2 \\
&\quad \leq \left(1 + \frac{3}{\bar{\lambda}^2} + 2 \left(2 + \bar{\delta}^2 \kappa^2 \right) \|F\|^2 \right) \left(1 + \bar{\delta} \right) [v]_n^2 . \qquad (10.8a)
\end{aligned}
$$

The other related inequality,

$$
\left\| F_x^{-1} v \right\|_{n+1}^2 < \left(1 + \frac{3}{\bar{\lambda}^2} + 2 \left(2 + \bar{\delta}^2 \kappa^2 \right) \left\| F^{-1} \right\|^2 \right) \left(1 + \bar{\delta} \right) [v]_n^2 . \qquad (10.9)
$$

is proven similarly. Finally, choose K_+ and K_- such that

$$
\begin{aligned}
\sqrt{\left(1 + \frac{3}{\bar{\lambda}^2} + 2 \left(2 + \bar{\delta}^2 \kappa^2 \right) \|F\|^2 \right) \left(1 + \bar{\delta} \right)} &< K_+, \\
\sqrt{\left(1 + \frac{3}{\bar{\lambda}^2} + 2 \left(2 + \bar{\delta}^2 \kappa^2 \right) \|F^{-1}\|^2 \right) \left(1 + \bar{\delta} \right)} &< K_-.
\end{aligned}
$$

It is important to note that K_+ and K_- can be chosen independently of n. With respect to the original norm we then have

$$|F_x v| \le \frac{K_+}{\sqrt{2}}|v| \le K_+|v|, \quad \text{and} \quad \left|F_x^{-1}v\right| \le \frac{K_-}{\sqrt{2}}|v| \le K_-|v|.$$

The above inequalities imply that, with respect to the n^{th} pre-adapted norm and the original norm, we have

$$\begin{aligned} |F_x v| &\le K_+|v|, & \left|F_x^{-1}v\right| &\le K_-|v|, \\ |F_x v| &\le K_+[v]_n, & \left|F_x^{-1}v\right| &\le K_-[v]_n, \quad \text{and} \\ [F_x v]_{n+1} &\le K_+[v]_n, & \left[\!\left[F_x^{-1}v\right]\!\right]_{n+1} &\le K_-[v]_n. \end{aligned}$$

Note that, since the ratio $\frac{[v]_n}{|v|}$ can grow κ^2-slowly, it is not likely that the pair of inequalities

$$[F_x v]_{n+1} \le K_+|v|, \quad \left[\!\left[F_x^{-1}v\right]\!\right]_{n+1} \le K_-|v|,$$

are true. *If* they were true then it would have been relatively easy to show that the F-invariant splitting associated to any weakly hyperbolic invariant set is C^{γ}_{κ} for some appropriate γ and k.

10.1.2 Step 2: The neighbourhoods U_n and C^∞ approximation

In this step we will show that we can build C^∞ metrics out of the pre-adapted metrics constructed in the previous step.

For each n we approximate the inner product constructed for $v \in T_xM$ for $x \in P_n$ by a C^∞ inner product defined on the whole of M. We choose this C^∞ inner product so that inequalities 10.3, 10.4, 10.5b, 10.6, 10.7, 10.8a, and 10.9 are all preserved.

More specifically, we approximate these inner products in the following way. Fix $n \ge 0$. By reviewing the proofs of the inequalities 10.3, 10.4, 10.5b, 10.6, 10.7, 10.8a, and 10.9, we note that there exists a constant $0 < k < 1$ for which

$$\begin{aligned} [F_x^s(v_s)]^2_{n+1} &\le k^4\tilde{\lambda}^2[v_s]^2_n, \\ [F_x^{-u}(v_u)]^2_{n+1} &\le k^4\tilde{\lambda}^2[v_u]^2_n, \end{aligned}$$

$$
\begin{aligned}
[v]_{n-1} &\leq k^4 [v]_n , \\
\max \{ \| [p^s_x] \|_n , \| [p^u_x] \|_n \} &\leq \frac{1}{k^4}, \\
|v| &\leq k^4 \sqrt{2} [v]_n , \\
[v]_n &\leq k^4 B_n |v| , \\
\left\| \left[p^u_x p^{s\pm}_x \right] \right\|_n &\leq k^4 \tilde{\delta}, \\
\left\| \left[p^s_x p^{u\pm}_x \right] \right\|_n &\leq k^4 \tilde{\delta}, \\
\left\| \left[p^{u\pm}_x p^s_x \right] \right\|_n &\leq k^4 \tilde{\delta}, \\
\left\| \left[p^{s\pm}_x p^u_x \right] \right\|_n &\leq k^4 \tilde{\delta}, \\
[F_x (v)]_{n+1} &\leq k^4 K_+ [v]_n , \\
\left\| \left[F^{-1}_x (v) \right] \right\|_{n+1} &\leq k^4 K_- [v]_n .
\end{aligned}
$$

Since the splitting is continuous over the closed compact set $P_n \subset M$, and moreover, since the section of the bi-linear forms of $T_{P_n}\mathfrak{A}$, $\langle\langle \cdot , \cdot \rangle\rangle_{(n,\cdot)}$, is defined by a finite sum of continuous sections, the section $\langle\langle \cdot , \cdot \rangle\rangle_{(n,\cdot)}$ is itself a continuous section of the bi-linear forms of $T_{P_n}\mathfrak{A}$. Since the set P_n is closed (compact) in M, we can extend this C^0 section of the bi-linear forms of $T_{P_n}\mathfrak{A}$ to a C^0 section, $\langle\langle \cdot , \cdot \rangle\rangle'''_{(n,\cdot)}$, of the bi-linear forms of TM. Now, with respect to the C^0 topology of bi-linear forms of TM, we approximate the extended C^0 section, $\langle\langle \cdot , \cdot \rangle\rangle'''_{(n,\cdot)}$, by a C^∞ section, $\langle\langle \cdot , \cdot \rangle\rangle''_{(n,\cdot)}$, such that

$$
k \leq \frac{\langle\langle v, \bar{v} \rangle\rangle'''_{(n,y)}}{\langle\langle v, \bar{v} \rangle\rangle''_{(n,y)}} \leq \frac{1}{k}
$$

for all $y \in M$ and $v, \bar{v} \in T_y M$. This implies that for all $x \in P_n$, $v \in T_x M$, $v_s \in E^s_x$, and all $v_u \in E^u_x$, all of the inequalities in the above list are satisfied with respect to the norm, $[\cdot]''_{(n,\cdot)}$, induced by the inner product $\langle\langle \cdot , \cdot \rangle\rangle''_{(n,\cdot)}$ and where the k^4 multiplier has been replaced by $k^2 < 1$.

Now choose an open neighbourhood U_n of $P_n \subset M$ for which inequalities 10.1, 10.8a, and 10.9 are all preserved. We assert that this can be done because $k^2 < 1$. Let $\bar{U}_n$ be another open neighbourhood of $P_n \subset M$ whose closure is contained in U_n. Then the pair of open sets U_n and $M \backslash \overline{\bar{U}_n}$ is an open cover of the manifold M. Let γ_{U_n} and γ_{O_n} be a partition of unity subordinate to this cover, where γ_{U_n} is the the element of this partition of unity which is subordinate to U_n, and γ_{O_n} is the element subordinate to $M \backslash \overline{\bar{U}_n}$. Define the n^{th} adapted inner product and norm as follows

$$\begin{aligned}
\langle\!\langle v, \bar{v} \rangle\!\rangle'_{(n,x)} &= \gamma_{U_n}(x) \langle\!\langle v, \bar{v} \rangle\!\rangle''_{(n,x)} + \gamma_{\bar{U}_n}(x) \langle v, \bar{v} \rangle_x , \\
[v]'_{(n,x)} &= \sqrt{\langle\!\langle v, v \rangle\!\rangle'_{(n,x)}} .
\end{aligned}$$

for all $v, \bar{v} \in T_x M$ and $x \in M$.

Since each of the inequalities 10.1, 10.8a, and 10.9 holds when the pre-adapted norm, $[\cdot]_n$, is replaced by the original norm, $|\cdot|$, these inequalities hold for n^{th} adapted norm, $[\cdot]_n'$, formed of the average of the n^{th} approximation to the extended pre-adapted norm, $[\cdot]''_m$, and the original Riemannian norm. Finally, to be definite, we note that the symbol, $[\cdot]_{(n,\cdot)}$, used in the statement of the theorem refers to the n^{th} adapted norm, $[\cdot]_{(n,\cdot)}'$, which we have just defined. ∎

The previous theorem shows that we can associate an adapted metric of $\boldsymbol{M}$ to any weakly pseudo-hyperbolic pseudo-orbit, $\mathfrak{A}$, of $\boldsymbol{M}$ with respect to which the eventual contraction (expansion) associated with the splittings of $\mathfrak{A}$ becomes immediate contraction (expansion). The next lemma states a partial converse.

Lemma 10.2 *Let $\boldsymbol{M}_o$ denote $\boldsymbol{M}$ together with the original metric of M on each M_n. Let $\boldsymbol{M}_*$ denote $\boldsymbol{M}$ together with an adapted metric which is $\frac{1}{\sqrt{2}}$-(B_n)-related to $\boldsymbol{M}_o$, and note that the sequence (B_n) is κ^2-slowly varying.*

If $\mathfrak{A}$ is a 1-λ-hyperbolic pseudo-orbit of $\boldsymbol{M}_*$ with h_0-hyperbolic blocks then $\mathfrak{A}$ is a κ^2-λ-hyperbolic pseudo-orbit of $\boldsymbol{M}_o$ with $h_0\sqrt{2}B_0$-hyperbolic blocks.

Proof: We note that, for all $x \in \boldsymbol{X}_n$, $v_s \in \boldsymbol{E}^s_x$ and $v_u \in \boldsymbol{E}^u_x$, the conditions

$$\begin{aligned}
[F_x^m(v_s)]_{n+1} &\leq h_0\lambda^m [v_s]_n \\
\left[F_x^{-m}(v_u)\right]_{n+1} &\leq h_0\lambda^m [v_u]_n \\
\max\{\|[p_x^s]\|_n, \|[p_x^u]\|_n\} &\leq h_0
\end{aligned}$$

together with the $\frac{1}{\sqrt{2}}$-(B_n)-relatedness of $\boldsymbol{M}_*$ to $\boldsymbol{M}_o$ imply that

$$\begin{aligned}
|F_x^m(v_s)| &\leq h_0\sqrt{2}B_n\lambda^m |v_s| \\
\left|F_x^{-m}(v_u)\right| &\leq h_0\sqrt{2}B_n\lambda^m |v_u| \\
\max\{\|p_x^s\|, \|p_x^u\|\} &\leq h_0\sqrt{2}B_n
\end{aligned}$$

Since the sequence (B_n) is κ^2-slowly varying, the result follows. ∎

10.2 Lifts, $\mathfrak{g}$, Lipschitz close to F

Lemma 10.3 *Consider a C^1 diffeomorphism f, an adapted metric on M, and an $\varepsilon > 0$. Then there exists a sequence $R_n = R(n,\varepsilon) > 0$ and a family of C^1 neighbourhoods $V_n = V(n,\varepsilon)$ of f such that for all n, if $g \in V_n$ and $\mathfrak{X}$ is any n-strong τ_M-pseudo orbit (τ_M with respect to the original metric of M), then we have*

$$\begin{aligned} Lip\left([F-\mathfrak{g}]\big|_{\triangle_{R_n} T\mathfrak{X}} \right) &\leq \varepsilon, \\ Lip\left(\left[F^{-1}-\mathfrak{g}^{-1}\right]\Big|_{\triangle_{R_n} T\mathfrak{X}} \right) &\leq \varepsilon. \end{aligned}$$

with respect to the adapted metric of M.

Proof: This is essentially the same proof as that used to prove lemma 9.1. Fix n, and let $n_{\pm} = n \pm 1$. We first want to show that there exists a C^1 neighbourhood, $\overline{V}_{(n,n_-)}$, of f and a positive constant $\overline{R}_{(n,n_-)}$ for which for all $(y,z) \in M \times M$ for which $d(f(y), z) \leq \tau_M$ and $u, v \in B_{\overline{R}_{(n,n_-)}}(0) \subset T_y M$ we have

$$\begin{aligned} \left|\mathfrak{g}_{z,y}(u) - \mathfrak{g}_{z,y}(v) - F_{z,y}(u-v)\right| &\leq \varepsilon\,|u-v| \quad \text{and} \\ \left|\mathfrak{g}_{z,y}^{-1}(u) - \mathfrak{g}_{z,y}^{-1}(v) - F_{z,y}^{-1}(u-v)\right| &\leq \varepsilon\,|u-v| \end{aligned}$$

We begin by considering $y \in M_n$ and $z \in M_{n_-}$ for which $d(f(y), z) \leq \tau_M$ with respect to the original metric of M and where M_n and M_{n_-} are M with the n^{th} and $n-1^{th}$ adapted metrics respectively. The map $\mathfrak{g}_{z,y} = \exp_z^{-1} g \exp_y$ is then defined on the ball $B_{\tau_M}(0)$ with respect to the original metric. For $v \in B_{\tau_M}(0)$ we have

$$\begin{aligned} &D_v\left(F_{z,y} - \exp_z^{-1} g \exp_y\right) \\ &\quad = D_v\left[D_{f(y)}\left(\exp_z^{-1}\right) D_y f - \exp_z^{-1} g \exp_y\right] \\ &\quad = D_{f(y)}\left(\exp_z^{-1}\right) D_y f - D_{g(\exp_y(v))}\left(\exp_z^{-1}\right) D_{\exp_y(v)}(g)\, D_v\left(\exp_y\right). \end{aligned}$$

This is defined and equal to zero for $g = f$ and $v = 0$. Since this derivative is continuous in both v and g with respect to the n^{th} and $n-1^{th}$ adapted metrics, and moreover since the set $\{(y,z) \in M \times M \mid d(f(y), z) \leq \tau_M\}$ is compact, we are done proving the first inequality relative to M_n and M_{n_-}, by continuity. Similar arguments prove the second inequality.

So far we have defined a C^1 neighbourhood $\overline{V}_{n,n_-}$ of f and a positive constant $\overline{R}_{n,n_-}$. Now apply the same argument to the two pairs of manifolds (M_n, M_n) and $\left(M_n, M_{n_+}\right)$ to obtain the C^1 neighbourhoods $\overline{V}_{(n,n)}$ and $\overline{V}_{(n,n_+)}$ of f and the positive constants $\overline{R}_{(n,n)}$ and $\overline{R}_{(n,n_+)}$. Now define $\overline{V}_n = \overline{V}_{(n,n_-)} \bigcap \overline{V}_{(n,n)} \bigcap \overline{V}_{(n,n_+)}$ and $\overline{R}_n = \min\left\{\overline{R}_{(n,n_-)}, \overline{R}_{(n,n)}, \overline{R}_{(n,n_+)}\right\}$.

Finally define $V_n = \bigcap_{i=0}^{n} \overline{V}_i$ and $\hat{R}_n = \inf_{i=0}^{n} \overline{R}_i$. ■

Lemma 10.4 *Consider a $C^{1+\gamma}$ diffeomorphism f, an adapted metric on $\boldsymbol{M}$, and an $\varepsilon > 0$. Then there exits a κ^2-slowly decreasing sequence $\tilde{R}_n = \tilde{R}(n,\varepsilon) > 0$ such that for any $\mathfrak{r}_M$-pseudo orbit, $\mathfrak{A}$. ($\mathfrak{r}_M$ with respect to the original metric of $\boldsymbol{M}$), we have*

$$Lip\left(\left[F - \mathfrak{f}\right]\Big|_{\Delta(\tilde{R}_n)T\mathfrak{M}}\right) \;\leq\; \varepsilon,$$
$$Lip\left(\left[F^{-1} - \mathfrak{f}^{-1}\right]\Big|_{\Delta(\tilde{R}_n)T\mathfrak{M}}\right) \;\leq\; \varepsilon.$$

with respect to the adapted metric of $\boldsymbol{M}$.

Proof: The body of the proof of Lemma 9.2 shows that there exists a constant C such that

$$\begin{aligned} \left|\mathfrak{f}_{x,y}(u) - \mathfrak{f}_{x,y}(w) - F_{x,y}(u-w)\right| &\leq C\tilde{R}^{\gamma}\left|u-w\right| \quad \text{and} \\ \left|\mathfrak{f}^{-1}_{x,y}(u) - \mathfrak{f}^{-1}_{x,y}(w) - F^{-1}_{x,y}(u-w)\right| &\leq C\tilde{R}^{\gamma}\left|u-w\right| \end{aligned}$$

Consider $x \in \boldsymbol{X}_n$ and $h(x) \in \boldsymbol{X}_m$ where $n-1 \leq m \leq n+1$. Then $B_m \leq \kappa^2 B_n$. Let

$$\tilde{R}_n = \frac{1}{\sqrt{2}}\left[\frac{\hat{\varepsilon}}{\sqrt{2}C\kappa^2 B_n}\right]^{\frac{1}{\gamma}}$$

where $\hat{\varepsilon} \leq \varepsilon$ has been chosen so that $\sqrt{2}\tilde{R}_0 \leq \mathfrak{r}_M$. If $[u]_n, [w]_n < \tilde{R}_n$ then we have

$$|u|, |w| \leq \min\left\{\left[\frac{\varepsilon}{\sqrt{2}C\kappa^2 B_n}\right]^{\frac{1}{\gamma}}, \mathfrak{r}_M\right\} \leq 1,$$

and hence,

$$\begin{aligned} &\left\|\left[\mathfrak{f}_{i\circ h(x)\hat{x}(x)}(u) - \mathfrak{f}_{i\circ h(x)\hat{x}(x)}(w) - F_{i\circ h(x)\hat{x}(x)}(u-w)\right]\right\|_m \\ &\quad\leq\; B_m\left|\mathfrak{f}_{i\circ h(x)\hat{x}(x)}(u) - \mathfrak{f}_{i\circ h(x)\hat{x}(x)}(w) - F_{i\circ h(x)\hat{x}(x)}(u-w)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \; CB_m \frac{\varepsilon}{\sqrt{2}C\kappa^2 B_n} |v - w| \\
&\leq \; \sqrt{2}CB_m \frac{\varepsilon}{\sqrt{2}C\kappa^2 B_n} [v - w]_n \\
&\leq \; \varepsilon [u - w]_n ,
\end{aligned}
$$

Similarly

$$\left\| f^{-1}_{\mathrm{sub}(x)\hat{s}(x)}(u) - f^{-1}_{\mathrm{sub}(x)\hat{s}(x)}(w) - F^{-1}_{\mathrm{sub}(x)\hat{s}(x)}(u - w) \right\|_m \leq \varepsilon [u - w]_n .$$

∎

Chapter 11

Existence of f-invariant splittings

In this chapter, we are interested in showing that associated to every pseudo-hyperbolic pseudo-orbit there is a unique F-invariant splitting which makes the pseudo-orbit hyperbolic. The proof of this fact is really a very easy application of the C^0_κ-Section theorem proven in Part II. Recall that this Section theorem applies to a "double bundle" structure which consists of a "contracting" bundle and bundle morphism which is "over" an "expanding" bundle and bundle morphism. In the current application of this theorem, the "contracting" bundle and bundle morphism are essentially provided by the tangent bundle and tangent bundle morphism of the pseudo-orbit itself. However the structure of the "expanding" bundle and bundle morphism which is "below" the "contracting" bundle and bundle morphism is not so obvious.

Given a pseudo-hyperbolic pseudo-orbit, $\mathfrak{A}$, choosing an appropriate "expanding" bundle and bundle morphism with which to show that there is a splitting which makes the pseudo-orbit hyperbolic is really very easy. The "expanding" bundle can be chosen to be the base space of the tangent bundle. The "expanding" bundle morphism can be chosen to be the base morphism of the tangent morphism. That is we can, rather trivially, use the following commuting diagram

$$\begin{tikzcd}
T\mathfrak{A} \arrow[r, "F"] \arrow[d, dash] & T\mathfrak{A} \arrow[d, dash] \\
X \arrow[r, "h"] \arrow[d, dash, "Id_X"'] & X \arrow[d, dash, "Id_X"'] \\
X \arrow[r, "h"] & X
\end{tikzcd}$$

However, by using this pair of bundles, since the fibres of the "lower" "expanding" bundle consist of single points, the content of the continuity statement of the Section theorems is vacuous. This means that if we were interested in showing that the unique splitting is Hölder continuous, (i.e. $C_*^{0+\gamma}$), we would have to choose our "lower" bundle more carefully. Fortunately, in our case, we can use lemma 9.3 to show that the unique splitting is C_*^0 continuous.

11.1 Existence of unique F-invariant splittings

Lemma 11.1 *Consider a $C^{1+\gamma}$ diffeomorphism f, and let M_* denote M together with an adapted metric of M. Fix $\lambda\kappa < \tilde{\lambda} < 1 \leq \kappa$, $1 < K_-$, $1 < K_+$, and $1 < h_0 < \tilde{h}_0 < \frac{1}{\tilde{\lambda}}$. Then there exists a positive δ such that for any δ-1-λ-pseudo-hyperbolic pseudo-orbit, $\mathfrak{A}$, of M_* and f with h_0-hyperbolic blocks, for which $\|[F_x]\|_{(h(x),x)} < K_+$ and $\left\|[F_x^{-1}]\right\|_{(h^{-1}(x),x)} < K_-$ for all $x \in X$, there exists a unique F-invariant splitting $E^s \oplus_X E^u : X \to S_{\tilde{h}_0}(T\mathfrak{A}) \subset G_X^1 M$ with respect to which $\mathfrak{A}$ is a 1-$\tilde{\lambda}$-hyperbolic pseudo-orbit with $\tilde{h}_0$-hyperbolic blocks. In particular the sections E^s and E^u of $G_X M$ are C_*^0 with respect to the adapted metric of M.*

Proof: The first few steps of this proof essentially mimic the proof of the uniformly hyperbolic version of this lemma given by either Hirsch and Pugh [HP70][Theorem 6.2] or Shub [Shu87][Corollary 5.19].

Let $k\left(\hat{\lambda},\hat{\delta}\right) = \frac{1+\hat{\delta}^2}{\left[\frac{1}{\hat{\lambda}}-\hat{\delta}\right]^2}$, then since $\lambda\kappa < 1$, we can choose $h_0, \hat{\lambda}, \delta, \hat{\delta} > 0$ for which

$$
\begin{aligned}
(h_0+\delta)\,\lambda &\leq \hat{\lambda} < 1, && (11.1a)\\
\delta\ \max\{K_+, K_-\} &\leq \hat{\delta}, && (11.1b)\\
\left[\frac{1+\hat{\delta}^2}{\left[\frac{1}{\hat{\lambda}}-\hat{\delta}\right]^2}\right]\kappa &= k\left(\hat{\lambda},\hat{\delta}\right)\kappa < 1, && (11.1c)\\
\hat{\lambda}+\hat{\delta} &\leq 1, && (11.1d)\\
\frac{1}{\frac{1}{\hat{\lambda}}-\hat{\delta}} &\leq 1, && (11.1e)\\
\frac{\hat{\delta}}{1-k\left(\hat{\lambda},\hat{\delta}\right)} &\leq \bar{\delta}, && (11.1f)\\
2\bar{\delta}h_0 &\leq 1, && (11.1g)\\
h_0\left[\frac{1+2\bar{\delta}h_0}{1-2\bar{\delta}h_0}\right] &\leq \bar{h}_0, && (11.1h)\\
\lambda\bar{h}_0^2 + \max\{K_+, K_-\}\,\bar{h}_0^3\bar{\delta} &\leq \hat{\lambda}, && (11.1i)
\end{aligned}
$$

Let $k = k\left(\hat{\lambda},\hat{\delta}\right)$.

11.1.1 Step 1: The general setup

Since $\mathfrak{A}$ is a pseudo-hyperbolic pseudo-orbit with h_0-hyperbolic blocks, the splitting of $T\mathfrak{A}$ over any hyperbolic block, $\boldsymbol{P}_n$, is continuous and moreover it is contained in $\mathcal{S}_{h_0}(T\mathfrak{A})$. Let

$$p_x^s : \boldsymbol{E}_x^s \oplus_{\boldsymbol{X}} \boldsymbol{E}_x^u \to \boldsymbol{E}_x^s, \quad p_x^u : \boldsymbol{E}_x^s \oplus_{\boldsymbol{X}} \boldsymbol{E}_x^u \to \boldsymbol{E}_x^u,$$

where $\boldsymbol{E}_x^s \oplus_{\boldsymbol{X}} \boldsymbol{E}_x^u = T_{\boldsymbol{X}}\boldsymbol{M}$. Recall that since $\mathfrak{A}$ is a δ-1-λ-pseudo hyperbolic pseudo-orbit, for each $x \in \boldsymbol{X}$, the images of the splittings $\boldsymbol{E}^s_{\boldsymbol{h}^\pm(x)} \oplus \boldsymbol{E}^u_{\boldsymbol{h}^\pm(x)}$ via the maps $\boldsymbol{F}^\mp\left(\boldsymbol{h}^\pm(s)\right)$ are δ close, with respect to the adapted metric, to the splitting $\boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u$. That is $d\left(\left(\boldsymbol{E}_x^{s\pm} \oplus \boldsymbol{E}_x^{u\pm}, \boldsymbol{E}_x^s \oplus \boldsymbol{E}_x^u\right)\right)_x \leq \delta$. Recall that this means that

$$\left\|\left[p_x^{s\pm}p_x^u\right]\right\|_x,\ \left\|\left[p_x^s p_x^{u\pm}\right]\right\|_x,\ \left\|\left[p_x^{u\pm}p_x^s\right]\right\|_x,\ \left\|\left[p_x^u p_x^{s\pm}\right]\right\|_x < \delta.$$

Note that this implies that

$$
\begin{aligned}
\left\|\left[p_x^u p_x^{u\pm}\right]\right\|_x &= \left\|\left[p_x^u\left(Id - p_x^{s\pm}\right)\right]\right\|_x\\
&\leq [p_x^u]_x + \left\|\left[p_x^u p_x^{s\pm}\right]\right\|_x\\
&\leq h_0 + \delta.
\end{aligned}
$$

Similar arguments also imply

$$\left\|\left[p_x^s p_x^{s\pm}\right]\right\|_x, \left\|\left[p_x^{u\pm} p_x^u\right]\right\|_x, \left\|\left[p_x^{s\pm} p_x^s\right]\right\|_x \leq h_0 + \delta.$$

Fix n and consider $x \in \mathbf{X}_n$. Then $\mathbf{h}(x) \in \mathbf{X}_m$ where $n-1 \leq m \leq n+1$. Then $\mathbf{F}_x$ maps $\mathbf{E}_x^s$ to $\mathbf{E}_{\mathbf{h}(x)}^{s-}$ and $\mathbf{E}_x^u$ to $\mathbf{E}_{\mathbf{h}(x)}^{u-}$, that is, $p_{\mathbf{h}(x)}^{s-}\mathbf{F}_x = \mathbf{F}_x p_x^s$, and $p_{\mathbf{h}(x)}^{u-}\mathbf{F}_x = \mathbf{F}_x p_x^u$. Similarly, $\mathbf{F}_{\mathbf{h}(x)}^{-1}$ maps $\mathbf{E}_{\mathbf{h}(x)}^s$ to $\mathbf{E}_x^{s+}$ and $\mathbf{E}_{\mathbf{h}(x)}^u$ to $\mathbf{E}_x^{u+}$, that is, $p_x^{s+}\mathbf{F}_{\mathbf{h}(x)}^{-1} = \mathbf{F}_{\mathbf{h}(x)}^{-1} p_{\mathbf{h}(x)}^s$, and $p_x^{u+}\mathbf{F}_{\mathbf{h}(x)}^{-1} = \mathbf{F}_{\mathbf{h}(x)}^{-1} p_{\mathbf{h}(x)}^u$.

Since $\mathbf{F}_x$ is linear, we can express $\mathbf{F}_x$ with respect to the splittings $\mathbf{E}_x^s \oplus \mathbf{E}_x^u$ and $\mathbf{E}_{\mathbf{h}(x)}^s \oplus \mathbf{E}_{\mathbf{h}(x)}^u$ as follows

$$\mathbf{F}_x = \begin{matrix} \mathbf{A}_x & \mathbf{B}_x \\ \mathbf{C}_x & \mathbf{D}_x \end{matrix}$$

where

$$\begin{aligned}
\mathbf{A}_x &= p_{\mathbf{h}(x)}^s \mathbf{F}_x p_x^s \in \mathrm{L}\left(\mathbf{E}_x^s, \mathbf{E}_{\mathbf{h}(x)}^s\right) \\
\mathbf{B}_x &= p_{\mathbf{h}(x)}^s \mathbf{F}_x p_x^u \in \mathrm{L}\left(\mathbf{E}_x^u, \mathbf{E}_{\mathbf{h}(x)}^s\right) \\
\mathbf{C}_x &= p_{\mathbf{h}(x)}^u \mathbf{F}_x p_x^s \in \mathrm{L}\left(\mathbf{E}_x^s, \mathbf{E}_{\mathbf{h}(x)}^u\right) \\
\mathbf{D}_x &= p_{\mathbf{h}(x)}^u \mathbf{F}_x p_x^u \in \mathrm{L}\left(\mathbf{E}_x^u, \mathbf{E}_{\mathbf{h}(x)}^u\right)
\end{aligned}$$

Using conditions 11.1a and 11.1b, we can estimate the norms of $\mathbf{A}_x$, $\mathbf{B}_x$, $\mathbf{C}_x$, and $\mathbf{D}_x^{-1}$ as follows

$$\begin{aligned}
&\|[\mathbf{A}_x]\|_{(n,m)} \\
&\quad = \left\|\left[p_{\mathbf{h}(x)}^s \mathbf{F}_x p_x^s\right]\right\|_{(n,m)} = \left\|\left[p_{\mathbf{h}(x)}^s p_{\mathbf{h}(x)}^{s-}\right]\right\|_{(m,m)} \left\|\left[\mathbf{F}_x\big|_{\mathbf{E}_x^s}\right]\right\|_{(n,m)} \leq (h_0 + \delta)\,\lambda \leq \hat{\lambda}, \\
&\|[\mathbf{B}_x]\|_{(n,m)} \\
&\quad = \left\|\left[p_{\mathbf{h}(x)}^s \mathbf{F}_x p_x^u\right]\right\|_{(n,m)} = \left\|\left[p_{\mathbf{h}(x)}^s p_{\mathbf{h}(x)}^{u-}\right]\right\|_{(n,m)} \|[\mathbf{F}_x]\|_{(n,n)} \leq \delta K_+ \leq \hat{\delta}, \\
&\|[\mathbf{C}_x]\|_{(n,m)} \\
&\quad = \left\|\left[p_{\mathbf{h}(x)}^u \mathbf{F}_x p_x^s\right]\right\|_{(n,m)} = \left\|\left[p_{\mathbf{h}(x)}^u p_{\mathbf{h}(x)}^{s-}\right]\right\|_{(n,m)} \|[\mathbf{F}_x]\|_{(n,n)} \leq \delta K_+ \leq \hat{\delta}, \\
&\left\|\left[\mathbf{D}_x^{-1}\right]\right\|_{(m,n)} \\
&\quad = \left\|\left[p_x^u \mathbf{F}_{\mathbf{h}(x)}^{-1} p_{\mathbf{h}(x)}^u\right]\right\|_{(m,n)} = \left\|\left[p_x^u p_x^{u+}\right]\right\|_{(m,m)} \left\|\left[\mathbf{F}_{\mathbf{h}(x)}^{-1}\big|_{\mathbf{E}_{\mathbf{h}(x)}^u}\right]\right\|_{(m,n)} \leq (h_0 + \delta)\,\lambda \leq \hat{\lambda}.
\end{aligned}$$

Since x was arbitrarily chosen, these estimates hold for all $x \in \mathbf{X}$.

11.1.2 Step 2: Construction of the C^0_κ fibre contraction

Define $\boldsymbol{L}^s$ ($\boldsymbol{L}^u$) to be the C^0_κ bundle, $\mathrm{L}(\boldsymbol{E}^u, \boldsymbol{E}^s)$ ($\mathrm{L}(\boldsymbol{E}^s, \boldsymbol{E}^u)$), of bundle maps from the bundle $\boldsymbol{E}^u$ ($\boldsymbol{E}^s$) to the bundle $\boldsymbol{E}^s$ ($\boldsymbol{E}^u$). Recall that the fibre of $\mathrm{L}(\boldsymbol{E}^u, \boldsymbol{E}^s)$ over the point x is simply $\mathrm{L}(\boldsymbol{E}^u_x, \boldsymbol{E}^s_x)$. Let $\boldsymbol{D}^s$ ($\boldsymbol{D}^u$) denote the unit disc subbundle of $\boldsymbol{L}^s$ ($\boldsymbol{L}^u$).

Our aim is to show that the following, rather formal, "stable" fibre bundle map of the unit disc subbundle $\boldsymbol{D}^s$ of $\boldsymbol{L}^s$ over $\boldsymbol{X}$ over $\boldsymbol{X}$:

$$\boldsymbol{F}^s(x, x, P) = \left(\boldsymbol{h}(x), \boldsymbol{h}(x), \Gamma^s_{F_x}(P)\right)$$

where

$$\Gamma^s_{F_x}(P) = [B_x + A_x P]\,[D_x + C_x P]^{-1}.$$

is a 0-fibre contraction over the trivial "fibre expansion" $\boldsymbol{h}_\xi$. Lemma 6.3 then implies that $\boldsymbol{F}^s$ has a unique $\boldsymbol{F}^s$-invariant section $\sigma_{\boldsymbol{F}^s}$ of the unit disc subbundle $\boldsymbol{D}^s$. Since this section is $\boldsymbol{F}^s$-invariant, for $x \in \boldsymbol{X}_{\sigma^n(\xi)}$, we know that $\boldsymbol{F}_x$ maps the graph of $\sigma_{\boldsymbol{F}^s}(x, x)$ into the graph of $\sigma_{\boldsymbol{F}^s}(\boldsymbol{h}(x), \boldsymbol{h}(x))$. Denote the graph of $\sigma_{\boldsymbol{F}^s}(x, x)$ by the symbol $\tilde{\boldsymbol{E}}^s_{(x,x)}$.

If the "unstable" fibre bundle map, $\boldsymbol{F}^u$, is defined in an analogous fashion as the "stable" fibre bundle map $\boldsymbol{F}^s$ was defined, then similar arguments could be used to show that there exists an $\boldsymbol{F}^u$-invariant section of the unstable bundle $\boldsymbol{D}^u$, $\sigma_{\boldsymbol{F}^u}$. As above denote the graph of $\sigma_{\boldsymbol{F}^u}(x, x)$ by $\tilde{\boldsymbol{E}}^u_{(x,x)}$. Since $\sigma_{\boldsymbol{F}^s}$ and $\sigma_{\boldsymbol{F}^u}$ are $\boldsymbol{F}^s$ and $\boldsymbol{F}^u$-invariant respectively, it is easy to show that the splitting, $T\mathfrak{A} = \tilde{\boldsymbol{E}}^s \oplus_{\boldsymbol{X}} \tilde{\boldsymbol{E}}^u$ is $\boldsymbol{F}$-invariant.

We begin by considering the stable foliation, the result for the unstable foliation is proven similarly. We must first determine the contraction constant of $\boldsymbol{F}^s$. Consider P and $\bar{P}$ in $\mathrm{L}(\boldsymbol{E}^u_x, \boldsymbol{E}^s_x)$ for some $x \in \boldsymbol{X}$. In order to estimate the contraction constant itself we will need a bound on $\left\|\left[[D_x + C_x P]^{-1}\right]\right\|_{(m,n)}$. Since the inverse of a lower bound of the norm of an operator is an upper bound for the norm of the inverse of the same operator and conversely, we note that if $\|[P]\|_n < 1$ then we have

$$\bar{\lambda}^{-1} - \bar{\delta} \le \|[D_x]\|_{(n,m)} - \|[C_x]\|_{(n,m)} \|[P]\|_n \le \|[D_x + C_x P]\|_{(n,m)}.$$

This implies that

$$\left\|\left[D_x + C_x P\right]^{-1}\right\|_{(m,n)} \leq \frac{1}{\frac{1}{\lambda} - \delta}.$$

If $\left\|\left[\tilde{P}\right]\right\|_n < 1$ as well, the same inequality holds for $\left\|\left[\left[D_x + C_x \tilde{P}\right]^{-1}\right]\right\|_{(m,n)}$.

Now, in order to estimate the contraction constant itself, note the last inequality of the proof of lemma 5.1, and consider

$$\begin{aligned}
&\left\|\left[\Gamma^{*}_{F_x}(P) - \Gamma^{*}_{F_x}\left(\tilde{P}\right)\right]\right\|_n \\
&\quad = \left\|\left[\left[B_x + A_x P\right]\left[D_x + C_x P\right]^{-1} - \left[B_x + A_x \tilde{P}\right]\left[D_x + C_x \tilde{P}\right]^{-1}\right]\right\|_n \\
&\quad \leq \left\|\left[\left[B_x + A_x P\right]\left[D_x + C_x P\right]^{-1} - \left[B_x + A_x \tilde{P}\right]\left[D_x + C_x P\right]^{-1}\right]\right\|_n \\
&\qquad + \left\|\left[\left[B_x + A_x \tilde{P}\right]\left[D_x + C_x P\right]^{-1} - \left[B_x + A_x \tilde{P}\right]\left[D_x + C_x \tilde{P}\right]^{-1}\right]\right\|_n \\
&\quad \leq \left\|A_x\right\|_{(n,m)} \left\|\left[P - \tilde{P}\right]\right\|_n \left\|\left[D_x + C_x P\right]^{-1}\right\|_{(m,n)} \\
&\qquad + \left[\left\|B_x\right\|_{(n,m)} + \left\|A_x\right\|_{(n,m)} \left\|\left[\tilde{P}\right]\right\|_n\right] \left\|\left[D_x + C_x P\right]^{-1}\right\|_{(m,n)} \\
&\qquad\quad \times \left\|\left[\left[D_x + C_x \tilde{P}\right]^{-1}\right]\right\|_{(m,n)} \left\|C_x\right\|_{(n,m)} \left\|\left[P - \tilde{P}\right]\right\|_n \\
&\quad \leq \left[\frac{\hat{\lambda}}{\frac{1}{\lambda} - \delta} + \hat{\delta}\frac{\hat{\lambda} + \hat{\delta}}{\left[\frac{1}{\lambda} - \delta\right]^2}\right] \left\|\left[P - \tilde{P}\right]\right\|_n \\
&\quad \leq \frac{1 + \hat{\delta}^2}{\left[\frac{1}{\lambda} - \hat{\delta}\right]^2} \left\|\left[P - \tilde{P}\right]\right\|_n \\
&\quad \leq k \left\|\left[P - \tilde{P}\right]\right\|_n .
\end{aligned}$$

Hence the contraction constant k_{F^*} of F^* is bounded above by k, which together with condition 11.1c in turn implies that

$$k_{F^*}\kappa \leq k\kappa < 1.$$

To finish showing that F^* is a fibre contraction we must show that it satisfies conditions $1, 2, 3$, and 4 of the definition of a fibre contraction (see page 63). Since h is a homeomorphism defined on the whole of X conditions 3, and 4 are automatically satisfied. Since $k_{F^*} \leq k < 1$ condition 1 is also satisfied. In order to verify condition 2 we need to show that for all $(x, x, P) \in L^*$ for which $[P]_x \leq 1$ we have $\left\|\left[\Gamma^{*}_{F_x}(P)\right]\right\|_{h(x)} \leq 1$. Hence we consider conditions 11.1d and 11.1e together with

$$\left\|\left[\Gamma^{*}_{F_x}(P)\right]\right\|_x = \left\|\left[B_x + A_x P\right]\left[D_x + C_x P\right]^{-1}\right\|_{h(x)}$$

$$
\begin{aligned}
&\leq \ [[B_x + A_x P]]_{h(x)} \left\|[(D_x + C_x P]^{-1}\right\|_{h(x)} \\
&\leq \ \frac{\hat{\lambda} + \hat{\delta}}{\frac{1}{\hat{\lambda}} - \hat{\delta}} \\
&\leq \ 1.
\end{aligned}
$$

11.1.3 Step 3: Applying the C^0_κ-Section Theorem

The previous step has shown that $\boldsymbol{F}^s$ is a C^0_κ 0-fibre contraction to which we can apply lemma 6.3 to conclude that the graph, $\bar{\boldsymbol{E}}^s$, of $\sigma_{\boldsymbol{F}^s}$ is $\boldsymbol{F}^s$ and hence $\boldsymbol{F}$-invariant. The proof that the graph, $\bar{\boldsymbol{E}}^u$, of $\sigma_{\boldsymbol{F}^u}$ is $\boldsymbol{F}^u$ and hence $\boldsymbol{F}$-invariant is similar.

In order to establish the hyperbolicity of the original pseudo-orbit with respect to the F-invariant splitting, $\bar{\boldsymbol{E}}^s \oplus_{\boldsymbol{X}} \bar{\boldsymbol{E}}^u$, we must determine firstly how non-degenerate the splitting is and secondly how close the splitting is to the original non-F-invariant splitting. To do this we must estimate the norm, $\|[\boldsymbol{F}^s(0_{L^s})]\|$, of the $\boldsymbol{F}^s$-image of the zero section of the fibre bundle $\boldsymbol{L}^s$. To do this, fix x in $\boldsymbol{X}$, recall condition 11.1a and consider

$$
\begin{aligned}
\left\|\left[\Gamma^s_{F_x}(0_{L^s})\right]\right\|_x &= \left\|\left[B_x D_x^{-1}\right]\right\|_x \\
&\leq \ [[B_x]]_{(h^{-1}(x),x)} \left\|\left[D_x^{-1}\right]\right\|_{(x,h^{-1}(x))} \\
&\leq \ \hat{\delta}\hat{\lambda} \leq \hat{\delta}.
\end{aligned}
$$

This implies that

$$\sup_{x \in \boldsymbol{X}} [\boldsymbol{F}^s(0_x)] \leq \hat{\delta}.$$

Similar arguments show that $\sup_{x\in\boldsymbol{X}} [\boldsymbol{F}^u(0_x)] \leq \hat{\delta}$.

Lemma 6.3 and condition 11.1f then implies that

$$
\begin{aligned}
d\left(\left(0_{L^s_x}, L_{\bar{E}^s_x}\right)\right)_x = \sup_{x\in\boldsymbol{X}} \left\|\left[L_{\bar{E}^s_x}\right]\right\| &\leq \ \frac{\hat{\delta}}{1-k} \leq \bar{\delta}, \quad \text{and} \\
d\left(\left(0_{L^u_x}, L_{\bar{E}^u_x}\right)\right)_x = \sup_{x\in\boldsymbol{X}} \left\|\left[L_{\bar{E}^u_x}\right]\right\| &\leq \ \frac{\hat{\delta}}{1-k} \leq \bar{\delta},
\end{aligned}
$$

where $L_{\bar{E}^s_x}$ $\left(L_{\bar{E}^u_x}\right)$ denotes the linear map from $\boldsymbol{E}^s_x$ $(\boldsymbol{E}^u_x)$ to $\boldsymbol{E}^u_x$ $(\boldsymbol{E}^s_x)$ which represents the stable (unstable) subspace of the F-invariant splitting of $T_x\mathfrak{A}$ given above, and $d\left(\left(0_{L^s_x}, L_{\bar{E}^s_x}\right)\right)_x$ $\left(d\left(\left(0_{L^u_x}, L_{\bar{E}^u_x}\right)\right)_x\right)$ denotes the distance between the zero map $0_{L^s_x}$ $\left(0_{L^u_x}\right)$ and the linear map $L_{\bar{E}^s_x}$ $\left(L_{\bar{E}^u_x}\right)$ with respect to the canonical distance metric of $L^s_x = \mathrm{L}(E^u_x, E^s_x)$ ($L^s_x = \mathrm{L}(E^u_x, E^s_x)$) with respect to the adapted metric of $\boldsymbol{M}$.

Since the original splitting is h_0-non-degenerate, this last pair of inequalities together with conditions 11.1g, and 11.1h and lemma 4.2 imply that, for any $x \in \boldsymbol{X}$, the F-invariant splitting above x is $\bar{h}_0$-non-degenerate. We can then use the upper bound of lemma 4.1 twice to imply that

$$\bar{d}_G\left(\left(\boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x, \bar{\boldsymbol{E}}^s_x \oplus \bar{\boldsymbol{E}}^u_x\right)\right)_x \leq \bar{h}_0\bar{\delta},$$

where $\bar{d}_G(\cdot,\cdot)$ denotes the product (max) metric of the double Grassmannian bundle $\mathbf{G}^2(T\mathfrak{A})$. We can then use statement 1 of lemma 4.3 to show that

$$d\left(\left(\bar{\boldsymbol{E}}^s_x \oplus \bar{\boldsymbol{E}}^u_x, \boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x\right)\right)_x \leq \bar{h}_0^3\bar{\delta}.$$

Recall that this means that

$$\max\left\{[p^s_n\bar{p}^u_n]_n,\ [\bar{p}^s_n p^u_n]_n,\ [p^u_n\bar{p}^s_n]_n,\ [\bar{p}^u_n p^s_n]_n\right\} < \bar{h}_0^3\bar{\delta}.$$

Since p^s_n, p^u_n, $\bar{p}^s_n$, and $\bar{p}^u_n$ are all projections in $S_{\bar{h}_0}(T\mathfrak{A})$ we know that

$$\max\left\{[p^s_n\bar{p}^s_n]_n,\ [\bar{p}^s_n p^s_n]_n,\ [p^u_n\bar{p}^u_n]_n,\ [\bar{p}^u_n p^u_n]_n\right\} \leq \bar{h}_0^2 < \infty.$$

Finally, since $\boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x$ is a direct sum, we know that $p^s_x + p^u_x = Id$.

Using condition 11.1i, we can now estimate the norm of $\boldsymbol{F}_x$ with respect to the F-invariant splitting, $\bar{\boldsymbol{E}}^s_x \oplus \bar{\boldsymbol{E}}^u_x$, as follows

$$\begin{aligned}
\left\|\left[\boldsymbol{F}_x|_{\bar{\boldsymbol{E}}^s_x}\right]\right\|_{(n,m)} &= \|[\boldsymbol{F}_x\bar{p}^s_x]\|_{(n,m)} \\
&\leq \left\|\left[\boldsymbol{F}_x|_{\boldsymbol{E}^s_x}\right]\right\|_{(n,m)} \|[p^s_x\bar{p}^s_x]\|_n \\
&\quad + \left\|\left[\boldsymbol{F}_x|_{\boldsymbol{E}^u_x}\right]\right\|_{(n,m)} \|[p^u_x\bar{p}^s_x]\|_n \\
&\leq \lambda\bar{h}_0^2 + K_+\bar{h}_0^3\bar{\delta} \leq \bar{\lambda}.
\end{aligned}$$

A similar argument shows that $\left\|\left[\boldsymbol{F}_x^{-1}|_{\bar{\boldsymbol{E}}^u_x}\right]\right\|_{(n,m)} \leq \bar{\lambda}$.

Since x was chosen arbitrarily, these estimates hold for all $x \in \boldsymbol{X}$. This means that $\mathfrak{A}$ is a 1-$\bar{\lambda}$-hyperbolic pseudo-orbit with $\bar{h}_0$-hyperbolic blocks with respect to the F-invariant splitting $T\mathfrak{A} = \bar{\boldsymbol{E}}^s \oplus_{\boldsymbol{X}} \bar{\boldsymbol{E}}^u$. ∎

Note that the proof of the previous lemma made no use of the continuity part of the C^0_u-Section theorem. We only need to know that there exists a (unique)

F-invariant splitting of $T\mathfrak{A}$ with respect to which $\mathfrak{A}$ is hyperbolic. We can then use lemma 9.3 to note that this splitting is both unique and C^0_*.

Chapter 12

Supporting neighbourhoods of a hyperbolic pseudo-orbit

In the theory of uniformly hyperbolic dynamical systems, we can show that any invariant set which is close to a uniformly hyperbolic invariant set is itself uniformly hyperbolic (see [HP70, Shu87]). The combination of the main lemma of the previous chapter with this next lemma obtains the same result for weakly hyperbolic *pseudo-orbits*.

Lemma 12.1 *Consider* $(h_0 + \delta)\lambda < \bar{\lambda} < 1$, $\delta < 1 < h_0 < \bar{h}_0, 1 < K_+ < \bar{K}_+$, *and* $1 < K_- < \bar{K}_-$. *Consider a closed* 1-λ*-hyperbolic invariant set,* $\mathfrak{A}$, *of* M *with* h_0*-hyperbolic blocks, and assume that* M *is equipped with a Riemannian metric which is adapted to the maximally shifted closure,* $\bar{\mathfrak{A}}$, *of* $\mathfrak{A}$. *In particular assume that* $[F_x(v)]_{h(x)} \leq K_+ [v]_x$ *and* $\left[\!\left[F_x^{-1}(v)\right]\!\right]_{h^{-1}(x)} \leq K_- [v]_x$ *for all* $x \in X$ *and* $v \in T_x\mathfrak{A}$.

Then there exists a neighbourhood U *of* $\bar{\mathbf{i}}\left(\bar{\mathfrak{A}}\right)$ *in* M *and a monotonically decreasing positive sequence* (α_n) *for which* any (α_n)*-pseudo orbit,* $\mathfrak{B}\left(X, \mathbf{h}, \mathbf{i}, \chi, f\right)$, *of* M *and* f *for which* $\bar{\mathbf{i}}\left(\bar{X}\right) \subset U$, *is a* δ-1-$\bar{\lambda}$*-pseudo hyperbolic pseudo-orbit of* M *with* $\bar{h}_0$*-hyperbolic blocks.*

Proof: We denote $\mathfrak{A}(\mathfrak{A}, k, X, h, i, \chi, f)$, and $\bar{\mathfrak{A}}\left(\mathfrak{A}, \bar{k}, \bar{X}, \bar{h}, \bar{i}, \bar{\chi}, \bar{f}\right)$, by $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ respectively.

We begin by extending the F-invariant h_0-non-degenerate splittings for the invariant set $\bar{\mathfrak{A}}$ to $\bar{h}_0$-non-degenerate splittings for a neighbourhood $\bar{U}$ of $\bar{\mathbf{i}}\left(\bar{X}\right)$ in

$\boldsymbol{M}$.

More precisely, for each $n \geq 0$, recall that $\boldsymbol{X}_n$ denotes the n^{th} set in the partition of $\boldsymbol{X}$, and that $\mathfrak{A}$ is a monotonic factor of $\bar{\mathfrak{A}}_0$ via the factor map, $k \circ \mathbf{E}_f$. Recall also that P_n denotes the n^{th} set in the gradation of X. Since $\boldsymbol{\mathfrak{A}}$ and hence both $\mathfrak{A}$ and $\boldsymbol{\mathfrak{A}}$ are closed pseudo-orbits, we know that the gradation $\{P_n\}_0^\infty$ is a collection of closed sets. Since $\boldsymbol{\mathfrak{A}}$ is 1-λ-hyperbolic, lemma 9.3 shows that the h_0-non-degenerate splitting of $T_{P_n}M$ is continuous, and so we can extend it to a continuous h_0-non-degenerate splitting of $T_{\tilde{U}_n}M$ for some open neighbourhood $\tilde{U}_n$ of $i(P_n)$ in M. Since $\boldsymbol{\mathfrak{A}}$ is aligned, and more over $k \circ \mathbf{E}_f\left(\boldsymbol{X}_n\right) \subset P_n$, we can lift this continuous extension from the neighbourhood $\tilde{U}_n$ of $i(P_n) \subset M$ to a continuous extension over the neighbourhood $Id_{(n,*)}\left(\tilde{U}_n\right)$ of $\mathbf{i}\left(\boldsymbol{X}_n\right) \subset M_n$. In order to simplify the notation, we will denote this lifted neighbourhood, $Id_{(n,*)}\left(\tilde{U}_n\right)$ by $\boldsymbol{\tilde{U}}_n$. Define the neighbourhood $\boldsymbol{\tilde{U}}$ to be the disjoint union of the $\boldsymbol{\tilde{U}}_n$, then $\boldsymbol{X} \subset \boldsymbol{\tilde{U}} \subset \boldsymbol{M}$.

Recall that $Id_{(m,n)}$ is the identification of M_n with M_m. With this notation we note that, since $\boldsymbol{\mathfrak{A}}$ is closed under all positive $\imath$-shifts, we know that $Id_{(n+1,n)}\left(\mathbf{i}\left(\boldsymbol{X}_n\right)\right) \subset \mathbf{i}\left(\boldsymbol{X}_{n+1}\right)$, and moreover $Id_{(n+j,n)}\left(\mathbf{i}\left(\boldsymbol{X}_n\right)\right) \subset \mathbf{i}\left(\boldsymbol{X}_{n+j}\right)$ for all positive j. Since $\boldsymbol{\mathfrak{A}}$ is factored over $\mathfrak{A}$ by the surjective function $k \circ \mathbf{E}_f : \boldsymbol{X} \to X$, and moreover both $\mathfrak{A}$ and $\boldsymbol{\mathfrak{A}}$ are invariant sets we know that $\boldsymbol{\mathfrak{A}}$ is an aligned hyperbolic pseudo-orbit. This means that the splitting of the tangent fibre above the point $x \in \mathbf{i}\left(\boldsymbol{X}_n\right) \subset M_n$ is the same as the splitting of the tangent fibre above the point $x \in \mathbf{i}\left(\boldsymbol{X}_m\right) \subset M_m$ for all $n \leq m$.

While the splittings above points in $\mathbf{i}\left(\boldsymbol{X}_n\right) \subset \mathbf{i}\left(\boldsymbol{X}_m\right)$ will be the same in both M_n and M_m, the extended splittings for points in

$$\begin{aligned} \left(\boldsymbol{\tilde{U}}_n \backslash \mathbf{i}\left(\boldsymbol{X}_n\right)\right) \bigcap Id_{(m,n)}\left(\boldsymbol{\tilde{U}}_m \backslash \mathbf{i}\left(\boldsymbol{X}_m\right)\right) &\subset M_n, \quad \text{and} \\ Id_{(n,m)}\left(\boldsymbol{\tilde{U}}_n \backslash \mathbf{i}\left(\boldsymbol{X}_n\right)\right) \bigcap \left(\boldsymbol{\tilde{U}}_m \backslash \mathbf{i}\left(\boldsymbol{X}_m\right)\right) &\subset M_m. \end{aligned}$$

may differ. However, for points which are sufficiently close to $\mathbf{i}\left(\boldsymbol{X}_n\right)$, the extended splittings in M_n and M_m will be close.

Since $\boldsymbol{\mathfrak{A}}$ is an invariant set, we know that $f\imath = ih$. Further more, since $\boldsymbol{\mathfrak{A}}$ is 1-λ-hyperbolic with h_0-hyperbolic blocks, we know that, for all $x \in \boldsymbol{X}$, the lift of f in the fibre $T_{\mathbf{i}(x)}\boldsymbol{M}$ satisfies

$$\boldsymbol{F}_x = \boldsymbol{F}_{ih(x),i(x)} = \boldsymbol{F}_{fi(x),i(x)} = D_{i(x)}f.$$

$$
\begin{aligned}
\left[F_x(v_s)\right]_{\mathbf{E}(x)} &\le \lambda\,[v_s]_x\,,\\
\left\|F_x^{-1}(v_u)\right\|_{\mathbf{E}^{-1}(x)} &\le \lambda\,[v_u]_x\,,\\
\max\left\{[p_x^s]_x\,,[p_x^u]_x\right\} &\le h_0,\\
\left[F_x(v)\right]_{\mathbf{E}(x)} &\le K_+\,[v]_x\,,\\
\left\|F_x^{-1}(v)\right\|_{\mathbf{E}^{-1}(x)} &\le K_-\,[v]_x\,.
\end{aligned}
$$

Fix $i,j \in \{-1,0,1\}$ and let $U_n^{(i,j)} \subset \bar{U}_n$ and $0 < \alpha_n^{(i,j)} \le \epsilon_M$ be such that for all $y \in U_n^{(i,j)}$, $z_+ \in M_{n+i}$, and $z_- \in M_{n+j}$ for which $d\left(\left(f(y),z_+\right)\right)_{n+i}$, $d\left(\left(f^{-1}(y),z_-\right)\right)_{n+j} \le \alpha_n^{(i,j)}$ we have

- $\left\|F_{z_+,y}(v_s)\right\|_{z_+} < \frac{\lambda}{h_0+\delta}\,[v_s]_y$,
- $\left\|F_{z_-,y}^{-1}(v_u)\right\|_{z_-} < \frac{\lambda}{h_0+\delta}\,[v_u]_y$,
- $\left\|F_{z_+,y}(v)\right\|_{z_+} < \bar{K}_+\,[v]_y$,
- $\left\|F_{z_-,y}^{-1}(v)\right\|_{z_-} < \bar{K}_-\,|v|$,
- the forward (backward) splittings are close to current splitting, that is

$$
\begin{aligned}
d\left(\left(E^s_{(n,y)} \oplus E^u_{(n,y)}, F_{z_+,y}^{-1}\left(E^s_{(n+i,z_+)}\right) \oplus F_{z_+,y}^{-1}\left(E^u_{(n+i,z_+)}\right)\right)\right)_n &\le \delta, \text{ and}\\
d\left(\left(E^s_{(n,y)} \oplus E^u_{(n,y)}, F_{z_-,y}\left(E^s_{(n+j,z_-)}\right) \oplus F_{z_-,y}\left(E^u_{(n+j,z_-)}\right)\right)\right)_n &\le \delta.
\end{aligned}
$$

For any fixed $i,j \in \{-1,0,1\}$ this can be done since the projections, $p_{E^s_{(n,x)}}$, $p_{E^u_{(n,x)}}$, Df, Df^{-1}, and the exp map are all C^0_*, and moreover, $\bar{U}_n$ is contained in a compact subset of M. Finally let

- $\mathbf{i}\left(X_n\right) \subset U_n = \bigcap_{i,j\in\{-1,0,1\}} U_n^{(i,j)}$,
- $U = \bigsqcup_{n=0}^{\infty} U_n$, and
- $\alpha_n = \inf_{i,j\in\{-1,0,1\}} \alpha_n^{(i,j)}$.

Now consider an (α_n)-pseudo orbit, $\mathfrak{B}\left(\mathfrak{B},\bar{k},\bar{X},\bar{\mathbf{k}},\bar{\mathbf{i}},\bar{\chi},\bar{f}\right)$, of M and f for which $\bar{\mathbf{i}}\left(\bar{X}\right) \subset U$. Our goal is to show that $\mathfrak{B}$ is a δ-1-λ-pseudo hyperbolic pseudo-orbit.

We begin by constructing the splitting of $T\mathfrak{B}$. Since $\bar{\mathbf{i}}\left(\bar{X}\right) \subset U \subset \bar{U}$ we can use the extended splittings defined in each U_n. For $x \in \bar{X}$ let $\bar{E}^s_x = E^s_{\bar{\mathbf{i}}(x)}$, $\bar{E}^u_x = E^u_{\bar{\mathbf{i}}(x)}$, and let $\tilde{p}^s_x$ and $\tilde{p}^u_x$ denote the projections onto the respective subspaces of $T_x\mathfrak{B}$. Since $\tilde{\imath}(x) \in \tilde{U}$ we know that

$$\max\{\|[\tilde{p}^s_x]\|_x, \|[\tilde{p}^u_x]\|_x\} \le \tilde{h}_0.$$

Now consider $0 \le n$, and $x \in \tilde{X}_n$, then

$$\tilde{h}(x) \in \tilde{X}_{n-1} \bigsqcup \tilde{X}_n \bigsqcup \tilde{X}_{n+1} \qquad \tilde{h}^{-1}(x) \in \tilde{X}_{n-1} \bigsqcup \tilde{X}_n \bigsqcup \tilde{X}_{n+1}.$$

We will consider the case in which $\tilde{h}^{\pm 1}(x) \in \tilde{X}_{n-1}$, all of the other cases are similar and will be left to the reader.

Since $\tilde{\imath}(x) \in U_n \subset U_n^{(-1,-1)} \subset \tilde{U}_n$ and moreover since $d\left(\tilde{f} \circ \tilde{\imath}(x), \tilde{\imath} \circ \tilde{h}(x)\right) \le \alpha_n \le \alpha_n^{(-1,-1)}$, we know that

$$\begin{aligned}
[F_x(v_s)]_{\tilde{h}(x)} = \left\|\left[F_{(\tilde{\imath}\circ\tilde{h}(x)\tilde{\jmath}(x))}(v_s)\right]\right\|_{(\tilde{\imath}\circ\tilde{h}(x))} &\le \tilde{\lambda}\,[v_s]_x, \\
\left\|\left[F_x^{-1}(v_u)\right]\right\|_{\tilde{h}^{-1}(x)} = \left\|\left[F^{-1}_{(\tilde{\imath}\circ\tilde{h}^{-1}(x)\tilde{\jmath}(x))}(v_u)\right]\right\|_{(\tilde{\imath}\circ\tilde{h}^{-1}(x))} &< \tilde{\lambda}\,[v_u]_x, \\
\max\{\|[\tilde{p}^s_x]\|_x, \|[\tilde{p}^u_x]\|_x\} &\le \tilde{h}_0, \\
d\left(\left(E^s_x \oplus E^u_x, F_x\left(E^s_{\tilde{h}^{-1}(x)}\right) \oplus F_x\left(E^u_{\tilde{h}^{-1}(x)}\right)\right)\right)_x &\le \delta, \quad \text{and} \\
d\left(\left(E^s_x \oplus E^u_x, F_x^{-1}\left(E^s_{\tilde{h}(x)}\right) \oplus F_x^{-1}\left(E^u_{\tilde{h}(x)}\right)\right)\right)_x &\le \delta.
\end{aligned}$$

This means that,

$$\begin{aligned}
\left\|\left[p^s_x p^{s+}_x\right]\right\|_x &= \left\|\left[p^s_x\left(Id - p^{u+}_x\right)\right]\right\|_x \\
&\le \|[p^s_x]\|_x + \left\|\left[p^s_x p^{u+}_x\right]\right\|_x \\
&\le h_0 + \delta.
\end{aligned}$$

Similarly, $\|[p^u_x p^{u+}_x]\|_x \le h_0 + \delta$. This in turn implies that

$$\begin{aligned}
\left\|\left[F^s_x\big|_{E^s_x}\right]\right\|_{(x,\tilde{h}(x))} &= \left\|\left[p^s_{\tilde{h}(x)} F_x p^s_x\right]\right\|_{(x,\tilde{h}(x))} \\
&\le \left\|\left[p^s_{\tilde{h}(x)} p^{s+}_{\tilde{h}(x)}\right]\right\|_{\tilde{h}(x)} \left\|\left[F_x\big|_{E^s_x}\right]\right\|_{(x,\tilde{h}(x))} \\
&\le (h_0 + \delta)\,\lambda < \tilde{\lambda}
\end{aligned}$$

where $\|[\cdot]\|_{(x,\tilde{h}(x))}$ denotes the operator norm of a map from the fiber $T_x\mathfrak{A}$ to the fiber $T_{\tilde{h}(x)}\mathfrak{A}$ taken with respect to the appropriate fiber norms. Similar arguments show that $\left\|\left[F^u_x\big|_{E^u_x}\right]\right\|_{(x,\tilde{h}^{-1}(x))} \le \tilde{\lambda}$. ∎

Chapter 13

Shadowing stable manifolds for pseudo-orbits

In this chapter we show that any uniformly hyperbolic pseudo-orbit of $\boldsymbol{M}$ and f, is shadowed by a unique uniformly hyperbolic invariant set of $\boldsymbol{M}$ and f and hence of M and f. When the pseudo-orbit is contained in a compact subset of $\boldsymbol{M}$ we can even find uniformly hyperbolic invariant sets of $\boldsymbol{M}$ and g for any g which is sufficiently close to f. In both cases we obtain the existence of as well as our estimates of the hyperbolicity of the invariant set from an application of the fibre bundle version of the Unstable Manifold Theorem proven in Part II. This means that this pair of Shadowing theorems are really Shadowing Stable Manifold Theorems.

The uniqueness part of this pair of shadowing theorems essentially states that there exists a κ^4-slowly decreasing region around any 1-λ-hyperbolic invariant set $\mathfrak{B}$ of $\boldsymbol{M}_{\bullet}$ and f within which there can not exist any other 1-λ-hyperbolic invariant set $\hat{\mathfrak{B}}$ of $\boldsymbol{M}_{\bullet}$ and f *which has the same dynamics as* $\mathfrak{B}$. Note that this "exclusion region" does not apply to 1-$\tilde{\lambda}$-hyperbolic invariant sets for $\lambda < \tilde{\lambda} < 1$ nor does it apply to invariant sets whose dynamics ever carries the orbits *outside* of the exclusion region. In particular this means that there can exist distinct invariant sets which are, to within some fixed level of physical accuracy, indistinguishable experimentally.

In principle the shadowing theorems should state that *any* invariant set $\hat{\mathfrak{B}}\left(\hat{\mathfrak{B}}\left(\hat{X}, \hat{h}, \hat{j}, \hat{\gamma}\right), \hat{k}, \boldsymbol{X}, \hat{\boldsymbol{h}}, \boldsymbol{j}, \boldsymbol{f}\right)$ which shadows the pseudo-orbit $\boldsymbol{\mathfrak{A}}(\mathfrak{A}(X, h, i, \chi), k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{f})$ is *conjugate* to the invariant set $\boldsymbol{\mathfrak{B}}(\mathfrak{B}(X, h, j, \bar{\chi}), k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{j}, \boldsymbol{f})$ constructed in the proof of the theorem. Note that this means that, in principle at least, the maps and spaces $\bar{k}$, $\bar{h}$, $\bar{\chi}$, $\bar{\boldsymbol{h}}$, $\bar{X}$, and $\bar{\boldsymbol{X}}$ could be different from the maps and spaces k, h, χ, $\boldsymbol{h}$, X, and $\boldsymbol{X}$. However the statement that the invariant set $\bar{\boldsymbol{\mathfrak{B}}}$ shadows the pseudo-orbit $\boldsymbol{\mathfrak{A}}$ tacitly implies that there is a pair of homeomorphisms from $\bar{X}$ to X and from $\bar{\boldsymbol{X}}$ to $\boldsymbol{X}$ respectively with respect to which the respective maps listed commute in the appropriate ways[1]. This means that in order to apply the uniqueness part of the shadowing theorems one must supply "half" of the structure required to show that $\bar{\boldsymbol{\mathfrak{B}}}$ is conjugate to $\boldsymbol{\mathfrak{B}}$. It is because of this that the shadowing theorems do not in fact mention conjugacies.

Finally the factor $\frac{1}{\sqrt{7B_n}}$ is technically necessary for the current proof but should not be really required. Since we use the exponential map associated with the original Riemannian metric to lift the diffeomorphism from the manifold into the tangent bundle, we must pass from the adapted metric through the original metric an back to the adapted metric. It is this passage through the original metric which requires the use of the additional factor $\frac{1}{\sqrt{7B_n}}$ in our proof.

13.1 Shadowing for weak pseudo-orbits

Theorem 13.1 *Consider a* C^r $(r \geq 1 + \gamma)$ *diffeomorphism,* f, *of a compact manifold* M. *Assume that the classifying manifold,* $\boldsymbol{M}$, *associated to* M *is equipped with the metric constructed in Theorem 10.1 which is* $\frac{1}{\sqrt{7}}$-(B_n)*-related to the original metric of* $\boldsymbol{M}$ *(and hence of* M*) where* (B_n) *is a* κ^2 *slowly increasing sequence. If* $0 < \lambda\kappa < \check{\lambda} < 1 < h_0 < \bar{h}_0$ *then there exists a pair of* κ^2*-slowly decreasing sequences* (α_n) *and* (R_n) *as well as a positive constant* K *such that,* $\alpha_n \leq R_n$ *and if* $0 < \dot{\rho} \leq 1$ *then*

- *associated to any* 1-λ*-hyperbolic* $(\dot{\rho}\alpha_n)$*-pseudo orbit* $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{f})$ *of* $\boldsymbol{M}$ *and* f *with* h_0*-hyperbolic blocks, there is a* 1-$\check{\lambda}$*-hyperbolic invariant set* $\bar{\boldsymbol{\mathfrak{B}}}(\bar{\mathfrak{B}}, \bar{k}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{h}}, \boldsymbol{j}, \boldsymbol{f})$ *of* $\boldsymbol{M}$ *and* f *with* $\bar{h}_0$*-hyperbolic blocks which*

[1]See chapter 8 for the relevant definitions

– $(KB_n\hat{\rho}\alpha_n)$*-shadows* $\mathfrak{A}$ *with respect to the adapted metric,* $\boldsymbol{M}_*$, *and*

– $(K\hat{\rho}\alpha_n)$*-shadows* $\mathfrak{A}$ *with respect to the original metric,* $\boldsymbol{M}_0$,

- *the invariant set* $\mathfrak{B}$ *has* $C_x^{r+\gamma}$ *stable and unstable manifolds,*
- *the sequence* $(B_n\alpha_n)$ *is a* $\kappa^{2/\gamma-2}$*-slowly* decreasing *sequence,*
- *if* $\hat{\mathfrak{B}}$ *is an invariant set of* M *and* f *which lifts to an invariant set* $\hat{\mathfrak{B}}\left(\mathfrak{B}, \hat{k}, \boldsymbol{X}, \boldsymbol{h}, \hat{j}, \boldsymbol{f}\right)$ *of* $\boldsymbol{M}$ *and* f *which* $\left(\frac{R_n}{B_n}\right)$*-shadows* $\mathfrak{A}$ *with respect to* $\boldsymbol{M}_o$ *(* $\left(\frac{R_n}{\sqrt{2}B_n}\right)$*-shadows* $\mathfrak{A}$ *with respect to* $\boldsymbol{M}_*$*) then* $\hat{j} = j$ *and hence* $\hat{\mathfrak{B}} = \mathfrak{B}$ *and* $\hat{\mathfrak{B}} = \mathfrak{B}$.

Proof: The proof of this theorem is a direct application of the perturbed unstable manifold theorem proven in Part II. Not surprisingly we prove this theorem in two steps. In the first step we verify the hypotheses needed to apply Theorem 7.1. In the second step we apply the theorem.

13.1.1 Verifying Hypotheses

Given the fact that $0 < \lambda\kappa < \hat{\lambda} < 1 < h_0 < \hat{h}_0$, fix the positive constants ε and δ as in theorem 7.1.

Since $\mathfrak{A}$ is 1-λ-hyperbolic with h_0-hyperbolic blocks, we know that there is an $\boldsymbol{F}$-invariant splitting of the tangent bundle of $\mathfrak{A}$, $T\mathfrak{A} = \boldsymbol{E}^s \oplus_{\boldsymbol{X}} \boldsymbol{E}^u$ for which

$$\left|\left|\left|\boldsymbol{F}|_{\boldsymbol{E}^s}\right|\right|\right| \leq \lambda, \quad \text{and} \quad \left|\left|\left|\boldsymbol{F}^{-1}|_{\boldsymbol{E}^u}\right|\right|\right| \leq \lambda.$$

Moreover, if $\{\boldsymbol{X}_n\}_0^\infty$ is the partition of $\boldsymbol{X}$ associated to $\mathfrak{A}$, then for all $x \in \boldsymbol{X}_n$, $\boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x \in \mathcal{S}_{h_0}(T_{\boldsymbol{X}_n}\boldsymbol{M}_n)$. This implies that for all $v = v_s + v_u \in T_x\mathfrak{A} = \boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x$ we have

$$\frac{1}{2}[v]_n \leq \max\{[v_s]_n, [v_u]_n\} \leq h_0 [v]_n.$$

Let $\left(\bar{R}_n\right)$ be the κ^2-slowly varying sequence from lemma 10.4. Recall that given a positive sequence such as $\left(\bar{R}_n\right)$, we can define a corresponding slowly varying function $\bar{R} : \boldsymbol{X} \to \mathbb{R}^+$ by $\bar{R}(x) = \bar{R}_{\chi(x)}$. We will assume that such a function has been defined for each of the sequences we will define below. Recall, also, that given a normed vector bundle and a slowly varying function we have defined slowly varying disc bundles (see page 23). With this notation, the previous inequality then implies that

$$\triangle_{\frac{\hat{R}}{2}} T\mathfrak{A} \subset \triangle_{\hat{R}} \boldsymbol{E}^s \oplus_{\boldsymbol{X}} \triangle_{\hat{R}} \boldsymbol{E}^u \subset \triangle_{h_0 \hat{R}} T\mathfrak{A}.$$

Let $R_n = \frac{\hat{R}_n}{4h_0}$, then, by lemma 10.4, we have that

$$\begin{aligned} Lip\left([F - \mathfrak{f}] \Big|_{\triangle_R \boldsymbol{E}^s \oplus_{\boldsymbol{X}} \triangle_R \boldsymbol{E}^u} \right) &\leq \varepsilon, \quad \text{and} \\ Lip\left(\left[F^{-1} - \mathfrak{f}^{-1}\right] \Big|_{\triangle_R \boldsymbol{E}^s \oplus_{\boldsymbol{X}} \triangle_R \boldsymbol{E}^u} \right) &\leq \varepsilon. \end{aligned}$$

Let

$$\alpha_n = \frac{\delta R_n}{2\sqrt{2}\kappa^2 K_-} \leq R_n$$

where the constant $\hat{\varepsilon} \leq \varepsilon$ used in the body of lemma 9.2 to define the sequence $\left(\hat{R}_n\right)$ has been made small enough to ensure that $\alpha_0 \leq \sqrt{2} B_0 \mathfrak{r}_M$.

Since $d(\boldsymbol{f} \circ \boldsymbol{i}(x), \boldsymbol{i} \circ \boldsymbol{h}(x)) \leq \hat{\rho}\alpha(\boldsymbol{h}(x))$ for all $x \in \boldsymbol{X}$, we have

$$d\left((\boldsymbol{f} \circ \boldsymbol{i}(x), \boldsymbol{i} \circ \boldsymbol{h}(x))\right)_{\boldsymbol{i}\circ\boldsymbol{h}(x)} \leq \frac{\delta R(\boldsymbol{h}(x))}{2\kappa^2} \leq \frac{\delta R(\boldsymbol{h}(x))}{2} \tag{13.1}$$

and hence that

$$d\left((\boldsymbol{f} \circ \boldsymbol{i}(x), \boldsymbol{i} \circ \boldsymbol{h}(x))\right)_{\boldsymbol{i}\circ\boldsymbol{h}(x)} \leq \frac{\delta R(x)}{2}.$$

This latter inequality implies that $\mathfrak{f}_x(0) \in B_{\delta R(x)}(0) \subset T_{\boldsymbol{h}(x)}\mathfrak{A}$ with respect to the box norm of the adapted metric of $\boldsymbol{M}$, this means that

$$\frac{[\![\mathfrak{f}_x(0_x)]\!]_{\boldsymbol{h}(x)}}{R(x)} \leq \delta. \tag{13.2}$$

Since $\delta < 1$, inequality 13.1 also implies that $\mathfrak{f}(\boldsymbol{0_X}) \subset \triangle_R \boldsymbol{E}^s \oplus_{\boldsymbol{X}} \triangle_R \boldsymbol{E}^u$.

Similarly, for all $x \in \boldsymbol{X}$, we have

$$\begin{aligned} d\left(\left(\boldsymbol{i}(x), \boldsymbol{f}^{-1} \circ \boldsymbol{i} \circ \boldsymbol{h}(x)\right)\right)_x &= d\left(\left(\boldsymbol{f}^{-1} \circ \boldsymbol{f} \circ \boldsymbol{i}(x), \boldsymbol{f}^{-1} \circ \boldsymbol{i} \circ \boldsymbol{h}(x)\right)\right)_x \\ &\leq K_- d\left((\boldsymbol{f} \circ \boldsymbol{i}(x), \boldsymbol{i} \circ \boldsymbol{h}(x))\right)_{\boldsymbol{h}(x)} \\ &\leq \frac{\delta R(\boldsymbol{h}(x))}{2\kappa^2} \\ &\leq \frac{\delta R(\boldsymbol{h}(x))}{2} \end{aligned} \tag{13.3a}$$

and hence that

$$d\left(\left(\boldsymbol{i}(x), \boldsymbol{f}^{-1} \circ \boldsymbol{i} \circ \boldsymbol{h}(x)\right)\right)_x \leq \frac{\delta R(x)}{2}.$$

This latter inequality implies that $\mathfrak{f}_x^{-1}(0) \in B_{\delta R(x)}(0) \subset T_{h^{-1}(x)}\mathfrak{M}$, again with respect to the box norm of the adapted metric, this means that

$$\frac{\left\|\left[\mathfrak{f}_x^{-1}(0_x)\right]\right\|_{h^{-1}(x)}}{R(x)} \leq \delta. \tag{13.4}$$

Again, since $\delta < 1$, inequality 13.3a also implies that $\mathfrak{f}^{-1}(0_X) \subset \triangle_R E^s \oplus_X \triangle_R E^u$. The two inequalities 13.2 and 13.4 together imply that

$$\rho = \max\left\{\frac{[\![\mathfrak{f}_x(0_x)]\!]_{h(x)}}{R(x)}, \frac{\left\|\left[\mathfrak{f}_x^{-1}(0_x)\right]\right\|_{h^{-1}(x)}}{R(x)}\right\} \leq \delta$$

where $[\![\mathfrak{f}_x(0_x)]\!]_{h(x)}$ and $\left\|\left[\mathfrak{f}_x^{-1}(0_x)\right]\right\|_{h^{-1}(x)}$ are both taken with respect to the box norm associated to the adapted metric of M.

13.1.2 Applying the perturbed Unstable manifold theorem

Since $\rho < \delta$ if we let $r_n = \frac{\rho}{\delta} R_n$ then, we know that $r_n \leq R_n \leq \frac{r_M}{2}$. The most important conclusion of Theorem 7.1 is that there exists a section, which we will denote as $\mathfrak{j}$, of the fibre bundle $\pi : X \to \triangle_r E^s \oplus_X \triangle_r E^u$, which is $\mathfrak{f}$-invariant.

Shadowing: Since $r_n \leq \frac{r_M}{2}$ we know that $\mathfrak{j}(X) \subset \triangle_{r_M} T\mathfrak{M}$ *with respect to the original metric of M.* This means that we can define a map $j : X \to M$ by $j(x) = \exp_{i(x)}(\mathfrak{j}(x))$.

Recall that $\mathfrak{f}$-invariance for the section, $\mathfrak{j}$, means that $\Gamma_{\mathfrak{f}}(\mathfrak{j}) = \mathfrak{j}$, that is $\mathfrak{f} o \mathfrak{j} o h^{-1} = \mathfrak{j}$. This means that, in each fibre $T_x\mathfrak{M}$ for all $x \in X$, we have $\mathfrak{j}(x) = \mathfrak{f}_x \circ \mathfrak{j} \circ h^{-1}(x)$, that is

$$\begin{aligned} \exp^{-1}_{i \circ h(x)} \circ f \circ \exp_{i(x)}(\mathfrak{j}(x)) &= \mathfrak{j}(h(x)), \\ f \circ \exp_{i(x)}(\mathfrak{j}(x)) &= \exp_{i \circ h(x)} \circ \mathfrak{j} \circ h(x), \\ f \circ j(x) &= j \circ h(x). \end{aligned}$$

Let $K = \frac{\kappa^2\sqrt{2}}{\delta}$. Since $\mathfrak{j}$ is a section of the fibre bundle $\pi : X \to \triangle_r E^s \oplus_X \triangle_r E^u$, we know that $[\![\mathfrak{j}(x)]\!]_{x,(\text{box norm})} \leq r(x)$ with respect to the box norm of the adapted metric. Since, $r(x) \leq \frac{\rho}{\delta} R(x)$, and $\rho \leq \frac{\hat{\rho}\alpha(x)\kappa^2}{R(x)}$, we know that

$$[\![\mathfrak{j}(x)]\!]_x \leq 2\,[\![\mathfrak{j}(x)]\!]_{x,(\text{box norm})} \leq 2r(x) \leq 2\frac{\hat{\rho}\alpha(x)\kappa^2}{\delta} \leq K(x)\frac{\hat{\rho}\alpha(x)}{\sqrt{2}}$$

Recall that $\exp_{i(x)}$ is the exponential map of the *original* Riemannian metric of $\boldsymbol{M}$. With respect to that metric we know that for all $x \in \boldsymbol{M}$ and $v \in B_{r_M}(0) \subset T_x\boldsymbol{M}$ we have

$$d(x, \exp_x(v)) = |v|_x .$$

Since the adapted metric of $\boldsymbol{M}$ is $\frac{1}{\sqrt{2}}(B_n)$-related to the original metric of $\boldsymbol{M}$, we know that, with respect to the original metric of $\boldsymbol{M}$, the previous inequality becomes

$$d\left(\boldsymbol{i}(x), \exp_{i(x)}(v)\right) \leq |v|_{i(x)} \leq \sqrt{2}\,[v]_{i(x)} .$$

This means that for $x \in \boldsymbol{X}$ we have

$$d(\boldsymbol{i}(x), \boldsymbol{j}(x)) \leq K\hat{\rho}\alpha(x) .$$

With respect to the adapted metric of $\boldsymbol{M}$, and for $x \in \boldsymbol{X}_n$ and $v \in B_{\frac{r_M}{\sqrt{2}}}(0) \subset T_{i(x)}\boldsymbol{M}$, these inequalities become

$$d\left(\left(\boldsymbol{i}(x), \exp_{i(x)}(v)\right)\right)_n \leq B_n d\left(\boldsymbol{i}(x), \exp_{i(x)}(v)\right) \leq \sqrt{2}B_n\,[v]_{i(x)} .$$

This means that for $x \in \boldsymbol{X}_n$ we have

$$d((\boldsymbol{i}(x), \boldsymbol{j}(x)))_n \leq KB(x)\,\hat{\rho}\alpha(x) .$$

These inequalities mean that the *unfactored* invariant set, $\mathfrak{B}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{j}, \boldsymbol{f})$, $(K\hat{\rho}\alpha_n)$-shadows $((KB_n\hat{\rho}\alpha_n)$-shadows) the *factored* $(\hat{\rho}\alpha_n)$-pseudo orbit $\mathfrak{A}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{f})$ with respect to original (adapted) metric $\boldsymbol{M}_o$ $(\boldsymbol{M}_a)$. Moreover, from section 8.4 of chapter 8, we know that any *unfactored* invariant set factors over an invariant set of M and f, and hence $\mathfrak{B}$ is a factored invariant set of $\boldsymbol{M}$ and f.

From the definitions of R_n, α_n, K_n given above, and the definition of R_n given in the body of lemma 9.2, it is easy to see that

$$B_n\alpha_n = \frac{1}{B_n^{\frac{1}{\gamma}-1}}\,\frac{\delta}{16h_0\kappa^2K_-}\left[\frac{\hat{\varepsilon}}{\sqrt{2}C\kappa^2}\right]^{\frac{1}{\gamma}} .$$

From the definition of B_n given in the body of theorem 10.1 we note that

$$\frac{B_{n+1}\alpha_{n+1}}{B_n\alpha_n} = \kappa^{2-\frac{2}{\gamma}} \leq 1 .$$

This implies that the sequence $(B_n\alpha_n)$ is a $\kappa^{\frac{2}{\gamma}-2}$-slowly decreasing sequence.

Uniqueness: Consider an invariant set $\mathfrak{B}$ of M and f which lifts to an invariant set $\tilde{\mathfrak{B}}\left(\mathfrak{B}, k, X, h, \tilde{j}, f\right)$ of $\boldsymbol{M}$ and f which (R_n)-shadows the pseudo-orbit $\mathfrak{A}$ with respect to the original metric on $\boldsymbol{M}$. Define $\tilde{\mathbf{j}}(x) = \exp^{-1}_{i(x)}\left(\tilde{j}(x)\right)$. In order to apply the uniqueness part of Theorem 7.1 we must show that $\left\|\tilde{\mathbf{i}}(x)\right\|_{x,(\text{box norm})} \leq R(x)$ with respect to the box norm of the adapted metric. Since the exponential map used in the definition of $\tilde{\mathbf{j}}$ is the exponential map associated with the *original* metric on $\boldsymbol{M}$ we must use the following inequality

$$d\left(x, \exp_x(v)\right) = |v|_x$$

in order to use the fact that $\tilde{\mathfrak{B}}$ shadows $\mathfrak{A}$.

Since $\tilde{\mathfrak{B}}$ $\left(\frac{R_n}{\sqrt{2}B_n}\right)$-shadows $\mathfrak{A}$ with respect to the adapted metric we know that

$$d\left(\left(i(x), \tilde{j}(x)\right)\right)_n \leq \frac{R(x)}{\sqrt{2}B(x)}$$

for all $x \in X_n$. Since the adapted metric is $\frac{1}{\sqrt{2}}$-(B_n)-related to the original metric, this inequality becomes

$$d\left(i(x), \tilde{j}(x)\right) \leq \sqrt{2}d\left(\left(i(x), \tilde{j}(x)\right)\right)_n \leq \frac{R(x)}{B(x)}$$

for all $x \in X_n$. That is the invariant set $\tilde{\mathfrak{B}}$ $\left(\frac{R_n}{B_n}\right)$-shadows the pseudo-orbit $\mathfrak{A}$ with respect to the original metric on $\boldsymbol{M}_o$.

This then implies that

$$\left\|\tilde{\mathbf{j}}(x)\right\|_{x,(\text{box norm})} \leq \left\|\tilde{\mathbf{j}}(x)\right\|_x \leq B(x)\left|\tilde{\mathbf{j}}(x)\right|_x \leq B(x)\, d\left(i(x), \tilde{j}(x)\right) \leq R(x).$$

From the inequalities used to show that $\mathfrak{B}$ shadows $\mathfrak{A}$ we know that

$$\|\mathbf{j}(x)\|_{x,(\text{box norm})} \leq r(x) \leq \frac{\rho}{\delta}R(x) \leq R(x).$$

That is $\mathbf{j}$ and $\tilde{\mathbf{j}}$ are *both* $\mathbf{f}$ invariant sections of the fibre bundle $\pi : \boldsymbol{X} \to \triangle_R\boldsymbol{E}^s \oplus_{\boldsymbol{X}} \triangle_R\boldsymbol{E}^u$. The uniqueness part of Theorem 7.1 then implies that $\mathbf{j}(x) = \tilde{\mathbf{j}}(x)$ and hence that $j(x) = \tilde{j}(x)$ for all $x \in \boldsymbol{X}$.

Finally, since (B_n) and (R_n) are κ^2-slowly *increasing* and *decreasing* sequences respectively, it is easy to show that $\left(\frac{R_n}{B_n}\right)$ is a κ^4-slowly *decreasing* sequence.

Hyperbolicity: Let $\tilde{F}_x = D_{\mathfrak{j}(x)}\mathfrak{f}$. Then $\tilde{F}$ is a bundle map of $T\mathfrak{A} \simeq T\mathfrak{B}$ where the isomorphism is given by the bundle map $D\mathfrak{j}\exp_i$ where the exp map is in this case the exponential map associated to the original metric.

The next most important conclusion of Theorem 7.1 is that the points $\mathfrak{j}(x)$ are hyperbolic with a hyperbolicity constant of $\tilde{\lambda}$ with respect to the fibre bundle map $\mathfrak{f}$. Recall that, in the context of theorem 7.1, this means that there exists a $\tilde{F}$-invariant splitting of $T\mathfrak{B} \simeq T\mathfrak{A} = \tilde{E}^s \oplus_{\boldsymbol{X}} \tilde{E}^u$ for which, for $x \in \boldsymbol{X}_n$ and $\boldsymbol{h}^{\pm}(x) \in \boldsymbol{X}_{m_{\pm}}$ $(n-1 \leq m_{\pm} \leq n+1)$ we have,

$$
\begin{aligned}
\left[\!\left[\tilde{F}_x(v_s)\right]\!\right]_{m_+} &\leq \tilde{\lambda}\,[v_s]_n, \\
\left[\!\left[\tilde{F}_x^{-1}(v_u)\right]\!\right]_{m_-} &\leq \tilde{\lambda}\,[v_u]_n, \\
\max\{\|[\tilde{p}_x^s]\|_n, \|[\tilde{p}_x^u]\|_n\} &\leq \tilde{h}_0,
\end{aligned}
$$

where $v_s \in \tilde{E}_x^s$, $v_u \in \tilde{E}_x^u$ and the projections $\tilde{p}_x^s$ and $\tilde{p}_x^u$ denote the projections of $T_x\mathfrak{B}$ onto the subspaces $\tilde{E}_x^s$ and $\tilde{E}_x^u$ respectively. Interpreted in the context of a pseudo-orbit, these conditions imply that the invariant set, $\mathfrak{B}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{j}, \boldsymbol{f})$, is 1-$\tilde{\lambda}$-hyperbolic with $\tilde{h}_0$-hyperbolic blocks.

Stable and Unstable manifolds: For any $x \in \boldsymbol{X}$, we can use the exponential map, $\exp_{i(x)}$, associated to the original metric of $\boldsymbol{M}$ to exponentiate the stable and unstable sections, g^s and g^u respectively, down to the manifold $\boldsymbol{M}$. Since the sections, g_x^s and g_x^u, are C^r, so are the exponentiated manifolds. Since the sections are continuous over each hyperbolic block, $\boldsymbol{P}_n$, so are the exponentiated manifolds. We leave the details to the reader. (For example, see Shub's version of this argument in [Shu87]). ∎

13.2 Shadowing for strong pseudo-orbits

Theorem 13.2 *Consider a C^r $(r \geq 1)$ diffeomorphism of a compact manifold M. Fix $0 < \lambda\kappa < \tilde{\lambda} < 1 < h_0 < \tilde{h}_0$. Then for each $n > 0$, there exists a C^1 neighbourhood, V_n, of f and positive constants, α_n and K_n, such that if $g \in V_n$ is a C^r diffeomorphism of M, $0 < \rho \leq 1$, and $\mathfrak{A}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, f)$ is a 1-λ-hyperbolic n-strong $\rho\alpha_n$-pseudo orbit of $\boldsymbol{M}$ and f with h_0-hyperbolic blocks, then there exists* *a unique 1-$\bar{\lambda}$-hyperbolic g-invariant set* $\mathfrak{B}(\boldsymbol{X}, \boldsymbol{h}, \boldsymbol{j}, \boldsymbol{g})$ *which* $K_n\bar{\rho}\alpha_n$*-shadows* $\mathfrak{A}$, *and moreover, the g-invariant set* $\mathfrak{B}$ *has* C^r_* *stable and unstable manifolds with respect to the diffeomorphism g.*

Proof: This is, not surprisingly, essentially the combination of theorems 6.2, 7.6, and 7.8 of [Shu87] (see also [HP70]).

Again, the proof of this theorem is a direct application of the perturbed Stable manifold theorem proven in Part II. It is essentially the same as the proof used to prove Theorem 13.1 above. The only difference is that we use lemma 10.3 instead of lemma 9.2. We also note that the n-strong pseudo-orbit $\mathfrak{A}$ is a *uniform* α_n-pseudo orbit, that is $d((\boldsymbol{f} \circ \boldsymbol{i}(x), \boldsymbol{i} \circ \boldsymbol{h}(x))) \leq \alpha_n$ for *all* $x \in \boldsymbol{X}$. This latter fact (slightly) simplifies the proof. We leave the details to the reader. ■

Part IV

Applications

Chapter 14

Applications of the Theory

The results contained in Part III were intentionally left as a suite of mix and match lemmas. This is because, in practice, they will usually be used in various different combinations to suit the application at hand. In this chapter we give two uses of these lemmas. For the first use of our theory we prove, the Weak Shadowing Stable Manifold Theorem. This theorem exhibits what is probably the most general use of our suit of lemmas. For the second use our theory we prove, Lemma 14.2 and Corollary 14.3. These results are improved versions of one of Katok's results [Kat80][Theorem 4.1].

14.1 The Weak Shadowing Stable Manifold Theorem

The Weak Shadowing Stable Manifold Theorem is quite literally a shadowing version of Pesin's Stable Manifold Theorem. While we could have broken the theorem up into at least three distinct parts (strong shadowing, weak shadowing, and stable manifold theory) we have chosen to keep the theorem as one *whole* in order to *stress* that the techniques used to prove the shadowing *also* provide the estimates required to prove the existence of stable manifolds.

Theorem 14.1 (Weak Shadowing Stable Manifold Theorem)
Setup: *Consider a C^r ($r \geq 1 + \gamma$) diffeomorphism f of a Riemannian manifold M. Let $\boldsymbol{M}$ denote the classifying manifold of M together with its original*

metric. Consider constants λ, $\bar{\lambda}$, κ, *and* $\bar{h}_0$, *for which* $0 < \lambda\kappa < \bar{\lambda} < \frac{1}{\kappa} \leq 1 \leq \bar{h}_0$. *Then there exists a constant* $B_0 > 0$, *a* κ^2*-slowly decreasing* $R_n > 0$, *a strictly decreasing sequence* $\alpha_n > 0$ *and a strictly increasing sequence* $K_n > 0$, *for which the following are true, with respect to the* original *metric*[1].

Shadowing part: *Consider any* κ-λ*-hyperbolic invariant set* $\mathfrak{H}$ *of* M *and* f. *Let* $\boldsymbol{\mathfrak{H}}$ *denote the minimally factored lift of* $\mathfrak{H}$ *into* $\boldsymbol{M}$. *Let* $\bar{\boldsymbol{\mathfrak{H}}}$ *denote the maximally shifted closure of* $\boldsymbol{\mathfrak{H}}$. *Then there exists a neighbourhood* $\boldsymbol{U}$ *of* $\bar{\boldsymbol{\mathfrak{H}}}$ *in* $\boldsymbol{M}$, *and a family of neighbourhoods* V_n *of* f *in* $Diff^r(M)$ *for which the following is true. Fix* $0 < \bar{\rho} < 1$. *We must now choose between one or other of the following two cases:*

Strong case: *Consider any* fixed N, *any* $g \in V_N$, *and any* N*-strong factored* $\bar{\rho}\alpha_N$*-pseudo orbit* $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{g})$ *of* $\boldsymbol{M}$ *and* g *for which* $\boldsymbol{i}(\boldsymbol{X}) \subset \boldsymbol{U}$. *(Note that* α_N *is a fixed constant in this case).*

or

Weak case: *Consider any factored* $(\bar{\rho}\alpha_n)$*-pseudo orbit* $\boldsymbol{\mathfrak{A}}(\mathfrak{A}, k, \boldsymbol{X}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{\chi}, \boldsymbol{g})$ *of* $\boldsymbol{M}$ *and* f *for which* $\boldsymbol{i}(\boldsymbol{X}) \subset \boldsymbol{U}$.

Let $\bar{f}$ *denote either* g *or* f *depending on whether or not* $\boldsymbol{\mathfrak{A}}$ *is a strong or weak pseudo-orbit.*

Then $\mathfrak{A}$ *is* $(K_n\bar{\rho}\alpha_n)$*-shadowed (with respect to* $\bar{f}$*) by a* κ^2-$\bar{\lambda}$*-hyperbolic invariant set,* $\boldsymbol{\mathfrak{B}}\left(\mathfrak{B}, \bar{k}, \boldsymbol{X}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{i}}, \boldsymbol{\chi}, \bar{\boldsymbol{f}}\right)$, *of* $\boldsymbol{M}$ *and* $\bar{f}$, *which has* $B_0\bar{h}_0$*-hyperbolic blocks. If* $\bar{\boldsymbol{\mathfrak{B}}}$ *is any other invariant set which* (R_n)*-shadows* $\boldsymbol{\mathfrak{A}}$ *with respect to* $\bar{f}$, *then* $\bar{\boldsymbol{\mathfrak{B}}}$ *and* $\boldsymbol{\mathfrak{B}}$ *are equal.*

Stable Manifold part: *Let* P_n *denote the* n^{th} *set in the gradation of* $\mathfrak{B}$. *For any* x *in* P_n *let* $\boldsymbol{E}^s_x \oplus \boldsymbol{E}^u_x$ *denote the splitting of the tangent space of* $\bar{\boldsymbol{i}}(x)$ *with respect to which* x *is* κ-λ*-hyperbolic, let* D^s *and* D^u *denote unit disks of the same dimensions as* $\boldsymbol{E}^s_x$ *and* $\boldsymbol{E}^u_x$ *respectively, and let* U_x *be a neighbourhood of* x *in* P_n. *Then*

[1]In the weak case, the sequences (α_n) and (K_n) can be chosen to be κ^2-slowly decreasing and increasing respectively, and moreover the sequence $(K_n\alpha_n)$ is a $\kappa^{2/\gamma-2}$-slowly *decreasing* sequence.

1. *there exist local stable $W^s_{R_n}(x)$ and unstable $W^u_{R_n}(x)$ manifolds which are C^r,*

2. *for $i \in \{s,u\}$ there exist embeddings $\Phi^i : U_x \times D^i \to M$ for which $\Phi^i(y,0) = y$ and $\Phi^i(y,D^i) = W^i_{R_n}(y)$ for all $y \in U_x$,*

3. *the local stable and unstable manifolds are tangent at $\bar{i}(x)$ to $\boldsymbol{E}^s_x$ and $\boldsymbol{E}^u_x$ respectively, and*

4. *if y and z are points in the local stable (unstable) manifolds of x, then $d(f^n(y), f^n(z)) \leq \tilde{h}h_0\kappa^{2|n|}\lambda^{|n|}d(y,z)$ for all $n \geq 0$ ($n \leq 0$).*

Proof: The proof of this theorem consists of the direct application of theorem 10.1, lemma 12.1, lemma 11.1, and one or other of the pair of theorems 13.1 and 13.2.

Fix λ_1, λ_2, and λ_3 such that

$$\lambda\kappa < \lambda_1\kappa < \lambda_2\kappa < \lambda_3\kappa < \tilde{\lambda}.$$

Now fix $h_{0,0}$, $h_{0,1}$, $h_{0,2}$, and $h_{0,3}$ such that

$$1 < h_{0,0} < h_{0,1} < h_{0,2} < h_{0,3} < \tilde{h}_0 < \frac{\lambda_3}{\lambda_2},$$

and such that $h_{0,1}\lambda_1 < \lambda_2$. Fix $\tilde{K}_+$ and $\tilde{K}_-$ such that $1 < K_+ < \tilde{K}_+$ and $1 < K_- < \tilde{K}_-$ where K_+ and K_- have been fixed by theorem 10.1 applied to the constants $\delta = 0$, λ, κ, $\tilde{h}_0 = h_{0,0}$, and $\tilde{\lambda} = \lambda_1\kappa$. Finally fix δ such that $(h_{0,1} + \delta)\lambda_1 < \lambda_2$ and which, moreover, satisfies lemma 11.1 with $\lambda = \lambda_2$, $\tilde{\lambda} = \lambda_3\kappa$, $h_0 = h_{0,2}$, $\tilde{h}_0 = h_{0,3}$, $K_+ = \tilde{K}_+$, and $K_- = \tilde{K}_-$.

We use theorem 10.1 to show that there is an adapted metric of $\boldsymbol{M}$ with respect to which the κ-λ-hyperbolic invariant set $\mathfrak{H}$ is 1-λ_1-hyperbolic with $h_{0,1}$-hyperbolic blocks. We use lemma 12.1 to obtain a neighbourhood, $\boldsymbol{U}$ of $\mathfrak{H}$ in $\boldsymbol{M}$, such that if $\mathfrak{A}$ is a pseudo-orbit for which $\boldsymbol{i}(\boldsymbol{X})$ is contained in $\boldsymbol{U}$ then $\mathfrak{A}$ is a δ-1-λ_2-pseudo hyperbolic pseudo-orbit with $h_{0,2}$-hyperbolic blocks. We use lemma 11.1 to show that $\boldsymbol{T}\mathfrak{A}$ has a splitting with respect to which $\mathfrak{A}$ is a 1-λ_3-hyperbolic pseudo-orbit with $h_{0,3}$-hyperbolic blocks. Finally we use either theorem 13.1 or theorem 13.2 to show that $\mathfrak{A}$ is uniquely shadowed by an invariant set, $\mathfrak{B}$, which is 1-$\tilde{\lambda}$-hyperbolic with $\tilde{h}_0$-hyperbolic blocks.

We leave the details to the reader. ■

In this last theorem, we call the hyperbolic invariant set $\mathfrak{H}$ the supporting pseudo-orbit since it is the hyperbolicity of $\mathfrak{H}$ which ensures that the pseudo-orbit $\mathfrak{A}$ is itself hyperbolic enough to be shadowed.

Note that 11.1, and either 13.1 or 13.2 can be used to show that any weak (strong) pseudo-orbit which is also pseudo-hyperbolic is shadowed by a unique invariant set. This implies that there are conditions which depend only on the diffeomorphism f, the manifold M, and the constants $0 < \lambda < 1$ which ensure that near any numerically calculated orbit which satisfies these conditions there is a λ-hyperbolic orbit. Again we leave the details to the reader.

14.2 Supports of f-invariant measures

We are now interested in sharpening some of the results obtained by Katok [Kat80]. In section 4 of Katok's paper, he essentially proves three things. Firstly, in Theorem 4.1, generalizes a result originally proven by Anosov for uniformly hyperbolic dynamical systems by showing that we can find a *hyperbolic* periodic orbit as close as desired to any point in the support of μ. Secondly, in Theorem 4.2 and Corollaries 4.1 and 4.2, he shows that if μ is non-atomic then we can find irreducible aperiodic shifts of finite type as close as we like to any *single* point in the support of μ. Finally, in Theorem 4.3 he shows that $\max\left\{0, \limsup_{n\to\infty} \frac{\ln P_n(f)}{n}\right\} \geq h_\mu(f)$ where $P_n(f)$ denotes the set of periodic orbits of f whose period is n.

The machinery that we now have at our disposal can easily strengthen Katok's Theorem 4.1. Our machinery *should* also be able to strengthen his Theorem 4.2 and Corollaries 4.1 and 4.2. Unfortunately, while it is easy to obtain a *uniformly hyperbolic* shift of finite type which is as close as we like to any arbitrary finite collection of points in the support of μ, showing that ergodicity of μ implies that this shift of finite type is irreducible and aperiodic is not obvious.

Since the strengthened version of Katok's Theorem 4.1 is both a relatively easy example of the use of our machinery *and* it is of interest in its own right, we will restrict ourselves to proving a modification of Katok's Theorem 4.1. We will leave possible strengthenings of the rest of Katok's theory for future work. Note that all of the work in Katok's paper is a consequence of his "Main Lemma" proven in his section 3. It is important to note that Katok's Main Lemma is now a simple consequence of the Weak Shadowing Stable Manifold Theorem proven above.

Recall that, if f is a diffeomorphism of a compact n dimensional manifold M, then the set of Lyapunov regular points has measure 1 with respect to any f-invariant Borel probability measure, μ (see [Kat80], [Pes77], and [Ose68] for the relevant proofs and definitions). In particular, if μ is an ergodic f-invariant Borel measure whose Lyapunov exponents lie *outside* the interval, $\left[\ln\left(\bar{\lambda}\right), -\ln\left(\bar{\lambda}\right)\right]$ for $0 < \bar{\lambda} < 1$, then, for $\bar{\lambda} \leq \lambda < \frac{1}{\kappa} < 1 \leq h_0$ and $0 \leq k \leq n$, the set $\bar{\Lambda}^k_{\lambda}$ has measure 1. Recall that, from section 9.1.2, the set $\bar{\Lambda}^k_{\lambda}$ is the set of all points in M which are κ-λ-hyperbolic with h_0-hyperbolic blocks for all $\lambda < \frac{1}{\kappa} < 1 \leq h_0$ and which have k-dimensional stable subspaces, $\boldsymbol{E}^s_x$.

Our strengthened version of Katok's Theorem 4.1 is

Lemma 14.2 *Fix an ergodic (weakly mixing) f-invariant Borel measure, μ, whose Lyapunov exponents lie outside the interval* $\left[\ln\left(\bar{\lambda}\right), -\ln\left(\bar{\lambda}\right)\right]$ *for some $0 < \bar{\lambda} < 1$. Fix $\varepsilon > 0$, and $\bar{\lambda} < \lambda < 1$, and consider any finite collection of distinct points, C, in the support of μ. Then there exists a uniformly λ-hyperbolic periodic orbit of f which comes within ε of each point in C.*

Proof: The proof of this theorem is based on Katok's proofs of his Theorems 4.1 and 4.2 [Kat80]. Katok's proof of his Theorem 4.1 with his application of his Main Lemma replaced by our Weak Shadowing Stable Manifold Theorem proves lemma 14.2 if C consists of a single point. We will prove lemma 14.2 in the case where the set C consists of two points. The proof for general finite sets C contained in the support of μ is similar and will be left to the reader.

Assume that $C = \{x, \bar{x}\} \subset \operatorname{supp}(\mu)$. Since x is in the support of μ, we can find numbers k, $\hat{\lambda}$, κ, and n such that

$$\begin{aligned} \mu\left(B_{\frac{\varepsilon}{4}}(x) \bigcap \Lambda^k_{\kappa, \hat{\lambda}, n}\right) &> 0, \quad \text{and} \\ \hat{\lambda}\kappa &< \lambda < 1. \end{aligned}$$

Since μ is weakly mixing and $\bar{x}$ is in the support of μ we know that there exists a number m such that

$$\mu\left(f^{m}\left(B_{\frac{\epsilon}{4}}(x)\right)\bigcap\Lambda^{k}_{n,\lambda,\kappa^{n+m}}\bigcap B_{\frac{\epsilon}{4}}(\bar{x})\right)>0.$$

From the Weak Shadowing Stable Manifold theorem, there exists sequences, (α_n) and (K_n) which are κ^2-slowly decreasing and increasing respectively for which any $(\bar{\rho}\alpha_n)$-pseudo orbit formed of orbit pieces of $\Lambda^{k}_{n,\lambda}$ is $(K_n\bar{\rho}\alpha_n)$-shadowed by a λ-hyperbolic invariant set.

Choose $0<\bar{\rho}<1$ such that $\bar{\rho}<\frac{\epsilon}{4\max\{K_n\alpha_n, K_{n+m}\alpha_{n+m}\}}$, and let B and $\bar{B}$ denote subsets of the respective intersections

$$\begin{aligned} B_{\frac{\epsilon}{4}}(x) \quad &\cap \quad \Lambda^{k}_{n,\lambda,\kappa^{n}}, \quad \text{and} \\ f^{m}\left(B_{\frac{\epsilon}{4}}(x)\right) \quad &\cap \quad \Lambda^{k}_{n,\lambda,\kappa^{n+m}}\bigcap B_{\frac{\epsilon}{4}}(\bar{x}) \end{aligned}$$

whose respective diameters are $\bar{\rho}\alpha_n$ and $\bar{\rho}\alpha_{n+m}$ for which $\mu(B)$ and $\mu\left(\bar{B}\right)$ are both strictly positive.

By weak mixing, there exists a positive constant $\bar{m}$ for which

$$\mu\left(f^{\bar{m}}\left(\bar{B}\right)\bigcap B\right)>0.$$

This means that there exists a point $\bar{y}$ in $\bar{B}$ such that $f^{\bar{m}}(\bar{y})$ is contained in B. Similarly by construction we know that

$$\mu\left(f^{m}(B)\bigcap\bar{B}\right)>0.$$

This means that there exists a point y in B such that $f^{m}(y)$ is contained in $\bar{B}$. Note that, also by construction, both y and $\bar{y}$ are contained in $\Lambda^{k}_{n,\lambda}$. The pseudo-orbit, $\left\{y, f(y), \cdot, f^{m-1}(y), \bar{y}, f(\bar{y}), \cdot, f^{\bar{m}-1}(\bar{y})\right\}$ is a finite $(\bar{\rho}\alpha_n)$-pseudo orbit built of orbit pieces of $\Lambda^{k}_{n,\lambda}$. The Weak Shadowing Stable Manifold theorem then implies that there exists a real $m+\bar{m}$-periodic orbit, z, of f which is λ-hyperbolic which $\frac{\epsilon}{4}$-shadows the pseudo-orbit formed of the iterates of y and $\bar{y}$. Again by construction we have

$$\begin{aligned} d(x,z) \quad &\leq \quad d(x,y)+d(y,z)<\frac{\epsilon}{2}, \\ d(\bar{x},f^{m}(z)) \quad &\leq \quad d(\bar{x},\bar{y})+d(\bar{y},f^{m}(z))<\frac{\epsilon}{2}. \end{aligned}$$

∎

As a very simple corollary of the previous lemma, we have

Corollary 14.3 (Supports of f-invariant measures) *Fix an ergodic f-invariant Borel measure, μ, whose Lyapunov exponents lie outside the interval* $\left[\ln\left(\bar{\lambda}\right), -\ln\left(\bar{\lambda}\right)\right]$ *for some* $0 < \bar{\lambda} < 1$. *Then for any* $\bar{\lambda} < \lambda < 1$, *the support of the measure* μ *is contained in the closure of the set of* λ*-hyperbolic periodic orbits of* f.

■

Bibliography

[AMR83] Ralph Abraham, Jerrold E. Marsden, and Tudor Ratiu. *Manifolds, Tensor Analysis, and Applications*, volume 2 of *Global Analysis, Pure and Applied*. Addison-Wesley, 1983.

[Ano70] D. V. Anosov. A certain class of invariant sets of smooth dynamical systems. *Izdanis Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev*, 2:39–45, 1970. In Russian.

[Bej90] Aurel Bejancu. *Finsler geometry and applications*. Ellis Horwood, 1990.

[Bow75] R. Bowen. Equilibrium states and the ergodic theory of anosov diffeomorphisms. *Springer Lecture Notes in Mathematics*, 470, 1975.

[Dom62] Peter Dombrowski. On the geometry of the tangent bundle. *Journal Fur Mathematik*, 210:73–88, 1962.

[FHY83] A. Fathi, M. R. Herman, and J. C. Yoccoz. A proof of pesin's stable manifold theorem. *Springer Lecture Notes in Mathematics*, 1007:177–215, 1983.

[Hel78] Sigurdur Helgason. *Differential Geometry, Lie groups, and Symmetric spaces*. Academic Press, 1978.

[Hir76] Morris W. Hirsch. *Differential Topology*, volume 33 of *Graduate texts in maths*. Springer-Verlag, 1976.

[HP70] M. Hirsch and C. Pugh. Stable manifolds and hyperbolic sets. In Shing-Shen Chern and Stephen Smale, editors, *Global Analysis*, volume XIV of *Symposium of Pure Mathematics*, pages 133–? American Mathematical Society, 1970.

[HPS77] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant Manifolds*, volume 583 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, 1977.

[Hus75] D. Husemoller. *Fiber Bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2nd edition, 1975.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publications of the IHES*, 51:137–306, 1980.

[Kat81] A. B. Katok. Dynamical systems with hyperbolic structure. *Translations of the American Mathematical Society*, 116:43–95, 1981. Translation of Ninth Math. Summer School (Kaciveli, 1971), Izdanie Inst. Mat. Akad. Nuak Ukrain. SSR, Kiev, 1972, pp. 125-211.

[KT92] T. Krueger and S. Troubetzkoy. Markov partitions and shadowing for non-uniformly hyperbolic systems with singularities. *Ergodic Theory and Dynamical Systems*, 1992. To appear in the September issue.

[Mat68] John N. Mather. Characterization of Anosov diffeomorphisms. *Indag. Math.*, 30:479–483. 1968. Also published in: Nederl. Akad. van Wetensch. Proc. Ser. A, Amsterdam **71**.

[Ose68] V. I. Oseledec. A multiplicative ergodic theorem. lyapunov characteristic numbers for dynamical systems. *Transactions of the Moscow Mathematical Society.*, 19:197–231. 1968.

[Pes76] Ja. B. Pesin. Families of invariant manifolds corresponding to nonzero characteristic exponents. *Mathematics of the USSR Izvestija*, 10(4-6):1261–1305, 1976.

[Pes77] Ja. B. Pesin. Characteristic lyapunov exponents and smooth ergodic theory. *Russian Mathematical Surveys (From Uspekhi Mat. Nauk)*, 32(4):55–114, 1977.

[PS89] Charles Pugh and Michael Shub. Ergodic attractors. *Transactions of the American Mathematical Society*, 312(1):1–54, March 1989.

[Run59] H. Rund. *Differential Geometry of Finsler Spaces.* Unknown journal!, 1959.

[Sas58] Shigeo Sasaki. On the differential geometry of tangent bundles of riemannian manifolds. *Tôhoku Math Journal*, 10:338–354, 1958.

[SGL90] D. Hulin S. Gallot and J. Lafontaine. *Riemannian Geometry.* Springer-Verlag, 2 edition, 1990.

[Shu87] Michael Shub. *Global Stability of Dynamical Systems.* Springer-Verlag, 1987. With the collaboration of Albert Fathi and Remi Langevin. Translation of: Stabilité globale des systèmes dynamiques, (Astérisque, Vol. 56), Société Mathématique de France, 1978. Translated by Joseph Christy.

Symbols

Index

D173490